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Abstract

This paper studies land-price dynamics in a general-equilibrium model with technology choice and a jointly provided setup facility. A high-return Leontief technology requires common operating arrangements, whereas a lower-return linear technology can be used without them. Firms pool a fixed endowment of managerial resources to prepare these arrangements. Maintaining a given level of support becomes more demanding as aggregate capital increases, and projects differ in the minimum support they require. Entrepreneurs choose the more profitable technology whenever it is feasible. Project requirements are learned after capital is installed and are independent of installed capital, so their heterogeneity gives rise to deterministic aggregate technology shares. We derive technology choices, wages, saving, and land prices within the same equilibrium. A uniform distribution of project requirements produces a continuous, three-branch piecewise linear price map. We construct economies in which this map is eventually expanding on period- $(2n + 3)$ Markov partitions, characterize its invariant probability densities, and establish their stability for a specified family of distributional perturbations. The source of aggregate instability is the interaction of capital accumulation, a common setup constraint, and heterogeneous adoption requirements, rather than misperception or aggregate shocks.

Keywords: Technology choice; Joint setup; Adoption requirements; Asset prices; Chaotic dynamics

JEL: C61, G12, O33

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1 Introduction

How can optimizing technology choices generate persistent and irregular movements in the price of a long-lived asset? This paper studies this question in an economy with productive capital and a fixed-supply land asset. The two production technologies have different operating returns and different setup requirements. A common organizational constraint changes which firms can use the high-return technology as aggregate capital varies. This changes the income available for saving and, through land-market clearing, the price of land.

Our starting point is the investment and asset-market structure of Hirano and Toda (2024, Section 6). Investors have logarithmic preferences, finance investment from their own wealth, and receive opportunities to invest in productive capital intermittently. Investors without such an opportunity hold land. We add technology choice and a worker sector. Workers consume their labor income and can use an outside technology when they are not employed by entrepreneurs. This makes it possible to derive investors' production income separately from workers' income.

The production side has three distinct ingredients. First, firms can share the preparatory work needed to use a Leontief technology. They pool managerial resources to establish common operating procedures, while retaining separate ownership of their capital and operating income. Second, maintaining a given level of these arrangements requires more preparatory effort in a larger economy. We represent this negative externality by a setup requirement proportional to installed aggregate capital. Third, projects differ in the minimum level of support they require. This is heterogeneity in adoption conditions, not in the operating production functions: once adopted, each of the two technologies is the same across firms. The differences in adoption requirements are an additional assumption; they do not follow from joint setup itself.

These ingredients have a direct economic interpretation. Common procedures for coordinating inputs, arranging interfaces, and adapting production to a shared operating environment need not be prepared independently by every firm. Joint preparation avoids duplication, but does not increase the total time and organizational resources available. More extensive economic activity makes this preparation more demanding. Some projects can operate with relatively simple arrangements, whereas others require more support. A firm chooses the high-return technology if the common arrangements meet its requirement and otherwise uses the linear technology. The requirement is specific to the installed project and cannot be avoided by reducing that project's operating scale.

The resulting feedback can make aggregate investor income non-monotone in capital. At low capital stocks, all projects can use the high-return technology, so additional capital raises income. At intermediate stocks, an increase in capital reduces the support that can be provided from the fixed resource endowment. The associated loss of high-return projects may outweigh the direct gain from having more capital. At high stocks, all projects use the linear technology, and income again increases with capital. A continuous distribution of project requirements translates these individual adoption decisions into a continuous aggregate map. No entrepreneur misperceives the state or chooses a lower-return feasible technology.

The paper is related to several literatures. The asset-market structure is connected to work on rational bubbles and their macroeconomic effects, including Blanchard and Watson (1982), Tirole (1985), Santos and Woodford (1997), Kocherlakota (2009), Martin and Ventura (2012), Farhi and Tirole (2012), Galí (2014, 2021), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Brunnermeier et al. (2020). Froot and Obstfeld (1991) and Ikeda and Shibata (1992, 1995) analyze fundamentals-dependent asset-price dynamics, while Hirano and Toda (2024, 2025) discuss bubble economics and bubble necessity. Our object here is the dynamics of land prices generated by the production side, rather than a claim that the bounded price paths constructed below contain a rational-bubble component.

Another line of research studies nonlinear asset-price dynamics through switching behavior and heterogeneous expectations, as in Day and Huang (1990) and Brock and Hommes (1997). The heterogeneity in this paper instead concerns actual project requirements. Consumption, saving, portfolio allocation, and technology adoption are optimizing decisions. Idiosyncratic requirements are drawn after capital installation; their cross-sectional aggregation is deterministic. Thus aggregate price fluctuations do not rely on exogenous aggregate shocks or errors in beliefs.

Our mechanism also relates to endogenous fluctuations driven by technology choice and financial or production constraints: Aghion et al. (1999), Iwaisako (2002), Matsuyama (2007), Asano et al. (2012), Matsuyama et al. (2018), Asano and Yokoo (2019), Umezaki and Yokoo (2019), and Asano et al. (2022, 2024). Horii et al. (2013) examine the distributional implications of technological shifts. In contrast to the preference-based wealth and substitution effects underlying Grandmont's (1985) endogenous competitive cycles, our nonlinearity arises from a change in the set of projects that can adopt the more productive technology. Logarithmic preferences leave the saving propensity unchanged; the adoption margin changes the income that is saved.

A uniform distribution of adoption requirements gives a particularly tractable specification. It makes the amount of capital operated with the Leontief technology piecewise linear in aggregate capital. After normalization, the land-price map belongs to the three-branch family analyzed in the technology-choice models of Zhang and Yokoo (2026a,b). We use their Markov constructions and Frobenius–Perron techniques, rather than claiming these constructions as new general dynamical results. Our contribution is to derive this family from a specified asset-market and joint-setup economy, establish a consistent mapping from economic parameters to the Markov restrictions, and interpret a finite-dimensional density-stability argument as a perturbation of project requirements. We give direct expansion calculations and an explicit economic example, so that the equilibrium restrictions and the dynamical conclusions can be checked together.

Section 2 describes joint setup and derives technology choices and wages. Section 3 derives saving and the general capital and land-price maps. Section 4 introduces a uniform distribution of project requirements and obtains the three-branch map. Section 5 studies Markov dynamics, invariant probability densities, and stability. Section 6 concludes. The Appendix provides the supporting equilibrium, parameter, and dynamical arguments.

2 Joint setup and technology choice

2.1 Investors, workers, and operating technologies

Time is discrete. There is one consumption good and a continuum of investors indexed by $i \in I = [0, 1]$, with total measure one. Investor i has preferences

$$E_0 \sum_{t=0}^{\infty} \beta^t \log c_{it}, \quad 0 < \beta < 1. \quad (1)$$

The supply of land is fixed at $S > 0$, and each unit pays a constant dividend $D > 0$ per period. Its date- t price is $P_t > 0$. One unit of investment at date t produces one unit of capital for use at date $t + 1$; a fraction $\delta \in [0, 1]$ depreciates after production. Investors cannot borrow or lend, and there is no market for claims on other investors' capital. At each saving date, a new capital-investment opportunity arrives with probability $\pi \in (0, 1)$, independently across investors, dates, and current wealth. An investor with an opportunity can invest in capital or land; one without it can invest only in land.

Each investor owns a potential production unit. The total measure of these units is fixed at one. A unit with installed capital can operate it in the current period; the

arrival of a new investment opportunity determines whether its owner can install capital for the next period. This distinguishes the fixed population of potential firms from the fraction π receiving new opportunities.

Workers supply a fixed amount of labor $\xi > 0$. They do not save or hold assets, and consume all their current income. Labor not employed by firms can be used in an outside technology with constant productivity $\alpha > 0$. Thus the reservation wage is α .

There are exactly two operating technologies. If the Leontief technology is available to project i , its output is

$$F_i^L(k_{it}, \ell_{it}) = (A + \alpha) \min\{k_{it}, \ell_{it}\}, \quad A > B > 0. \quad (2)$$

The linear technology is always available and produces

$$F_i^R(k_{it}) = Bk_{it}. \quad (3)$$

Both technologies have the same depreciation rate. An entrepreneur operates one technology in each project. Installed capital remains tied to its project until current production is completed, after which its undepreciated portion can be converted one-for-one into the consumption good and its value included in current wealth. All capital used in the following period must be installed through a new investment opportunity, including capital financed from this recovered value. Operating output is retained by the owner of that capital.

The two functions are convenient limiting cases of differences in input substitutability. Autor and Dorn (2013) and Karabarbounis and Neiman (2014) motivate the relevance of labor-saving production, while Chirinko (2008) and Duffy and Papageorgiou (2000) discuss evidence on capital–labor substitutability. We use the two technologies as analytical specifications, not as literal descriptions of a particular industry. The setup requirements introduced next are separate from these operating functions.

2.2 A shared setup facility and its economic interpretation

Using the Leontief technology requires preparatory arrangements that firms can make jointly. Examples of the type of work we have in mind include establishing common input-coordination procedures, specifying interfaces between equipment, and adapting operating rules to a shared trading and logistical environment. The outcome of this work can be used by several firms without each repeating the same preparation. Joint

setup does not involve pooling productive capital, redistributing operating profits, or creating additional financial claims.

Each potential firm has one unit of managerial resources available for this preparatory work each period. Since the measure of firms is one, aggregate resources are also one. These resources are a separate in-kind endowment: their availability does not depend on the arrival of a capital-investment opportunity. They have no alternative remunerative use, generate no separate utility cost when used, and cannot be augmented by purchasing more units. Thus setting up the facility uses managerial resources, not additional consumption goods.

Let $u_t \geq 0$ be the level of support that the common arrangements can provide. The resources required to provide this level are

$$\Gamma(K_t, u_t) = K_t u_t, \quad K_t = \int_I k_{it} di > 0. \quad (4)$$

The aggregate stock K_t is predetermined when setup takes place. Equation (4) captures the assumption that maintaining a given support level becomes more demanding as the scale of the surrounding production environment increases. Installed capital contributes to this environment whether its owner subsequently chooses the Leontief or the linear technology. The linear technology does not itself require the detailed coordination supplied by the facility. An individual firm takes this aggregate environment as given, and cannot reduce the common setup burden by changing only its own operating scale.

The proportional form in (4) is a technological externality assumption, not a consequence of fixing the number of firms. Similarly, the common use of completed arrangements is an assumption about the setup output: the managerial input is scarce, whereas the resulting procedures can be shared. The facility is prepared anew for the equipment and operating configuration of each period, and is not an additional accumulated state variable.

We consider a joint-setup arrangement in which a firm contributes its managerial endowment in exchange for access to the facility for that period. The consortium supplies the highest support level allowed by its members' resources. With full participation this gives

$$u_t = \frac{1}{K_t}. \quad (5)$$

Participation and provision are considered at the production stage, conditional on already installed capital. Under the labor-surplus condition below, more support can only expand a member's profitable production opportunities and does not raise the

wage. Since contribution has no other cost, full participation is weakly optimal. The consortium does not coordinate its members' earlier capital investments.

A separate group with a fraction $M \leq 1$ of the economy's managerial resources can provide at most M/K_t using the same setup technology. Such a group cannot make a project feasible when the economy-wide facility cannot support it. We use the full-participation arrangement, without requiring uniqueness of all possible participation outcomes.

2.3 Heterogeneous project requirements and timing

Project i at date t needs a minimum support level $\nu_{it} > 0$ to operate the Leontief technology. It can adopt that technology if and only if

$$\nu_{it} \leq u_t = \frac{1}{K_t}. \quad (6)$$

The index i denotes an entrepreneur's project, not a type of technology. All firms have the same operating functions (2)–(3). What varies across projects is the level of preparation needed to use the Leontief function. For example, one equipment configuration may fit relatively simple common procedures, whereas another needs more extensive arrangements. Once the necessary preparation is available, both configurations have the same operating productivity and input proportions in the model.

This heterogeneity is an additional assumption, distinct from sharing the setup burden. The coefficient ν_{it} is neither a membership payment nor the firm's contribution to aggregate setup costs. The common input requirement is $K_t u_t$; the economy does not pay a separate cost $\nu_{it} K_t$ for every project. The condition that preparation must meet a minimum level before positive production is possible is also separate from the Leontief operating function. In particular, an unsupported project cannot circumvent (6) by operating at a smaller scale.

Capital k_{it} is installed at date $t - 1$. At date t , the common facility is prepared and each project's requirement is learned before technology choice and production. Requirements are independently renewed across projects and dates, with a known continuous distribution function F_ν on $(0, \infty)$, and are independent of installed capital and past asset holdings. They describe project-specific configuration requirements rather than permanent entrepreneurial ability. After a requirement is learned, neither installed equipment nor the favorable requirement of another project can be reassigned to that

project. Entrepreneurs then observe their actual requirement and the market wage and choose a feasible technology. Saving and the allocation of wealth to next period's assets take place after current production.

We assume exact cross-sectional aggregation for the idiosyncratic draws, including aggregation weighted by installed capital and by current wealth. Specifically, almost surely at every date,

$$\begin{aligned}\int_I k_{it} \mathbf{1}_{\{\nu_{it} \leq u_t\}} di &= F_\nu(u_t) K_t, \\ \int_I \eta_{it} w_{it} di &= \pi W_t,\end{aligned}$$

where η_{it} indicates the arrival of a new capital-investment opportunity at the date- t saving stage, w_{it} is wealth immediately before that draw, and $W_t = \int_I w_{it} di$. These weighted aggregation identities are imposed in addition to the stated independence and timing assumptions. Together, these assumptions allow aggregate dynamics to be expressed without a separate distributional state variable. There are no perception errors or incorrect wage forecasts.

2.4 Production and the labor market

At the maximum support level, the capital installed in projects that can use the Leontief technology is

$$K_t^E = K_t F_\nu(1/K_t). \quad (7)$$

Let $q(K) := F_\nu(1/K)$. We focus on states satisfying

$$Kq(K) < \xi. \quad (8)$$

Section 4 gives a simple parameter restriction that makes this condition hold for every $K > 0$.

Proposition 1 (Technology choice and the production equilibrium). *Suppose (8) holds. Under the joint-setup arrangement, full participation, support $u = 1/K$, and wage $w = \alpha$ form a production-stage equilibrium. Projects satisfying $\nu_i \leq 1/K$ use the Leontief technology, while the remaining projects use the linear technology. Aggregate capital employed with each technology is*

$$K^L(K) = Kq(K), \quad K^R(K) = K[1 - q(K)], \quad (9)$$

and production income accruing to investors is $BK + (A - B)K^L(K)$.

Proof. The outside technology prevents the wage from falling below α . At any wage above α , all labor would be supplied to firms, but their demand cannot exceed the capital in eligible projects, $Kq(K) < \xi$. Hence such a wage cannot clear the market and $w = \alpha$.

At this wage an eligible project hires $\ell_i = k_i$ and earns $(A + \alpha)k_i - \alpha k_i = Ak_i > Bk_i$. An ineligible project can only use the linear technology. The capital-weighted aggregation assumption gives (9), and residual labor $\xi - K^L$ uses the outside technology. The stated investor income follows by adding the two sectors' wage-net output.

At any lower feasible support level the set of eligible projects is smaller, so labor remains in surplus and the wage remains α . Raising support therefore never reduces a firm's operating income. Since contribution uses an otherwise unremunerated resource, membership is weakly optimal and maximum provision is consistent with members' interests. A nonmember or a smaller separate consortium cannot obtain a higher support level with the same aggregate capital and setup technology. Thus there is no profitable unilateral change in participation or technology at the proposed allocation. $\square$

The reason a firm uses the lower-return technology is consequently not an erroneous comparison of aggregate marginal products. Every firm that can use the Leontief technology already does so. The other firms cannot meet its adoption requirement, even at the maximum support level.

Workers' total income and consumption are $C_t^W = \alpha\xi$. Writing Y_t for current production excluding land dividends, we obtain

$$\begin{aligned} Y_t &= (A + \alpha)K_t^L + B(K_t - K_t^L) + \alpha(\xi - K_t^L) \\ &= BK_t + (A - B)K_t^L + \alpha\xi. \end{aligned} \tag{10}$$

Thus $Y_t - C_t^W = BK_t + (A - B)K_t^L$. The managerial input is already accounted for by the in-kind setup constraint; there is no additional goods cost to subtract from this income.

3 Saving and land-price dynamics

3.1 Consumption, saving, and asset allocation

The individual budget constraint is

$$c_{it} + k_{i,t+1} + P_t s_{it} = Z_{it} k_{it} + (P_t + D) s_{i,t-1} =: w_{it}, \quad (11)$$

where s_{it} is land held at the end of date t , w_{it} is individual wealth, and the gross return on installed capital is

$$Z_{it} = B + 1 - \delta + (A - B) \mathbf{1}_{\{\nu_{it} \leq 1/K_t\}}. \quad (12)$$

The indicator equals one when the displayed condition holds and zero otherwise. The date- t wage, denoted w_t without an individual subscript, is α in the production equilibrium.

For a price-taking investor, both asset payoffs and feasible investment amounts are homogeneous in personal wealth. Setup access involves no monetary fixed charge, and adoption requirements do not depend on the investor's own investment amount. Logarithmic utility therefore gives

$$c_{it} = (1 - \beta) w_{it}, \quad C_t = (1 - \beta) W_t, \quad (13)$$

where C_t and W_t denote aggregate investor consumption and wealth. The consumption decision separates from the return-dependent part of the portfolio problem. Appendix A supplies the argument and verifies equilibrium on the capital paths used below.

Aggregating (11), using land-market clearing and Proposition 1, gives

$$W_t = (B + 1 - \delta) K_t + (A - B) K_t^L + (P_t + D) S, \quad (14)$$

$$K_{t+1} + P_t S = \beta W_t. \quad (15)$$

Heterogeneous realized project returns change the distribution of wealth but not the common saving propensity. By the wealth-weighted aggregation assumption in Section 2.3, investors receiving a new capital-investment opportunity hold aggregate wealth πW_t .

Write

$$R_t := \frac{P_{t+1} + D}{P_t} < B + 1 - \delta. \quad (16)$$

When this inequality holds along the candidate aggregate path, capital dominates land even in the worst project outcome. Investors with an investment opportunity put all their saving into capital, while the others hold land. Hence

$$K_{t+1} = \beta\pi W_t, \quad P_t S = \beta(1 - \pi)W_t. \quad (17)$$

This portfolio conclusion uses (16), not logarithmic utility alone. We verify the inequality on the same paths generated by the resulting map.

3.2 The general aggregate map

Define the coefficients

$$\begin{aligned} d_0 &:= 1 - \beta(1 - \pi) > 0, \\ a &:= \frac{\beta\pi(A + 1 - \delta)}{d_0}, \quad b := \frac{\beta\pi(B + 1 - \delta)}{d_0}, \quad c := \frac{\beta\pi DS}{d_0}. \end{aligned} \quad (18)$$

Here c without subscripts is the intercept of the aggregate map; c_{it} continues to denote individual consumption. Eliminating the land value in (14) with (17) yields

$$K_{t+1} = H(K_t) := bK_t + (a - b)K_t F_\nu(1/K_t) + c. \quad (19)$$

Land prices and next period's capital satisfy

$$P_t = \frac{1 - \pi}{\pi S} K_{t+1}. \quad (20)$$

Consequently, with $\chi := \pi S/(1 - \pi)$,

$$P_{t+1} = \frac{1}{\chi} H(\chi P_t) = \left[b + (a - b)F_\nu(1/(\chi P_t)) \right] P_t + \frac{c}{\chi}. \quad (21)$$

This price law follows from technology-specific operating income, saving, and market clearing. It is not an average of prices from two different economies.

The shape of (19) depends on how rapidly high-return projects become ineligible when capital increases. If F_ν has density f_ν , then, putting $v = 1/K$,

$$H'(K) = b + (a - b) \left[F_\nu(v) - v f_\nu(v) \right]. \quad (22)$$

Additional capital directly increases production, but also lowers the common support

level and removes some projects from the high-return technology. A decreasing part of the map arises when this adoption effect is strong enough:

$$vf_\nu(v) - F_\nu(v) > \frac{b}{a-b}. \quad (23)$$

A continuous distribution alone does not guarantee a decreasing branch or chaos.

3.3 A sufficient condition for an equilibrium path

Let $J = [\underline{K}, \overline{K}] \subset (0, \infty)$ be an invariant interval for H , meaning $H(J) \subseteq J$. Suppose that (8) holds on J . From (20), with $K = K_{t+1}$,

$$R_t = \frac{H(K) + \pi DS / (1 - \pi)}{K}. \quad (24)$$

Since $0 \leq F_\nu \leq 1$, we have $H(K) \leq aK + c$. Therefore a convenient sufficient condition for (16) is

$$a + \frac{\pi DS}{d_0(1 - \pi)\underline{K}} < B + 1 - \delta. \quad (25)$$

Lemma 1 (Equilibrium on an invariant interval). *Suppose that H has an invariant interval $J \subset (0, \infty)$, (8) holds on J , and (25) is satisfied. For every initial $K_0 \in J$, with nonnegative initial capital and land holdings having these aggregate supplies, positive individual wealth, and initial project requirements independent of installed capital, there is an equilibrium with full participation in joint setup and full capital investment by investors receiving an investment opportunity. Its aggregate dynamics is (19), and its land prices satisfy (20).*

Proof. See Appendix A. □

The equilibrium concept combines individual optimal saving and asset allocation, production-stage technology and labor choices, the specified membership and provision rule for joint setup, and goods and land-market clearing. Expectations of aggregate prices are correct. The joint-setup organization does not internalize the effect of aggregate investment on its future resource requirement. The lemma constructs equilibrium for initial capital in J ; it does not require global convergence to J from every possible initial allocation.

4 Adoption heterogeneity and the three-branch map

4.1 Uniform project requirements

To obtain a tractable specification, take two capital thresholds $0 < K_L < K_H$ and assume

$$\nu_{it} \sim \text{Unif} \left[\frac{1}{K_H}, \frac{1}{K_L} \right]. \quad (26)$$

The lower threshold K_L is the largest capital stock at which every project can adopt the Leontief technology. At K_H , almost no project can do so. These thresholds are determined by adoption requirements, not by the labor endowment ξ . Figure 2 illustrates the distribution.

When $K_L < K < K_H$, direct evaluation of the cumulative distribution gives

$$q(K) = \frac{1/K - 1/K_H}{1/K_L - 1/K_H} = \frac{K_L(K_H - K)}{(K_H - K_L)K}. \quad (27)$$

Thus the capital used with the Leontief technology is

$$K^L(K) = \begin{cases} K, & 0 < K \leq K_L, \\ \frac{K_L(K_H - K)}{K_H - K_L}, & K_L < K < K_H, \\ 0, & K \geq K_H. \end{cases} \quad (28)$$

In particular, $\sup_{K>0} K^L(K) = K_L$. The simple restriction

$$\xi > K_L \quad (29)$$

therefore ensures surplus labor and wage α in every state. No selection among multiple labor-clearing wages is needed at either adoption threshold.

Uniformity is used for analytical tractability. It makes $K^L(K)$, rather than the adoption share $q(K)$ itself, affine on the transition interval. Joint provision and finite resources do not imply this distribution. A common requirement shared by all firms would instead make adoption jump when support crosses that requirement. Continuous heterogeneity is what spreads the switch across an interval of capital stocks.

4.2 Capital dynamics and the invariant interval

Substituting (28) into (19) gives

$$H(K) = \begin{cases} aK + c, & K \leq K_L, \\ \left[b - \frac{(a-b)K_L}{K_H - K_L} \right] K + \frac{(a-b)K_L K_H}{K_H - K_L} + c, & K_L < K < K_H, \\ bK + c, & K \geq K_H. \end{cases} \quad (30)$$

The three branches meet continuously. The middle slope is

$$H'_M = \frac{bK_H - aK_L}{K_H - K_L}. \quad (31)$$

It is negative when $aK_L > bK_H$. Thus a sufficiently concentrated range of adoption requirements can produce a sharp enough decline in high-return capital to reverse the direct effect of capital accumulation.

We focus on $a > 1 > b > 0$ and define

$$\underline{K} := bK_H + c, \quad \overline{K} := aK_L + c. \quad (32)$$

Assume

$$0 < \underline{K} < K_L < K_H < \overline{K}. \quad (33)$$

Then $J = [\underline{K}, \overline{K}]$ is an invariant interval. The middle branch maps $[K_L, K_H]$ onto J . On the left branch $H(K) > K$, so its image of $[\underline{K}, K_L]$ stays in J . On the right, (33) implies $c < (1-b)K_H$, so $H(K) < K$ for $K \geq K_H$; its image of $[K_H, \overline{K}]$ also stays in J . Figure 1 shows this geometry.

4.3 The normalized land-price map

Let

$$\begin{aligned} \Delta &:= \overline{K} - \underline{K}, \\ \theta_L &:= \frac{K_L - \underline{K}}{\Delta}, \quad \theta_R := \frac{K_H - \underline{K}}{\Delta}, \\ x_t &:= \frac{\pi SP_t / (1 - \pi) - \underline{K}}{\Delta} = \frac{K_{t+1} - \underline{K}}{\Delta}. \end{aligned} \quad (34)$$

Under (33), $0 < \theta_L < \theta_R < 1$. Equations (30) and (20) give the map $\varphi : [0, 1] \rightarrow [0, 1]$,

$$x_{t+1} = \varphi(x_t) = \begin{cases} 1 + a(x_t - \theta_L) =: \varphi_L(x_t), & 0 \leq x_t \leq \theta_L, \\ \frac{\theta_R - x_t}{\theta_R - \theta_L} =: \varphi_M(x_t), & \theta_L < x_t \leq \theta_R, \\ b(x_t - \theta_R) =: \varphi_R(x_t), & \theta_R < x_t \leq 1. \end{cases} \quad (35)$$

This is a continuous N-shaped map: the two outer slopes are positive and the middle slope is negative. Its shape is derived from actual adoption opportunities, not from averaging perceived states.

The four normalized parameters $(a, b, \theta_L, \theta_R)$ cannot simply be assigned independently of the economic environment. Appendix B gives an explicit mapping back to $(\beta, \pi, \delta, A, B, D, S, K_L, K_H, \xi)$ and verifies the labor and asset-return restrictions. The next section uses this mapping to construct Markov economies, including an explicit period-5 example.

5 Markov dynamics and invariant probability densities

The map (35) belongs to the piecewise linear family used in the technology-choice models of Zhang and Yokoo (2026a,b). We first state the relevant definitions, then obtain the Markov restrictions directly in terms of the adoption thresholds.

5.1 Markov partitions and eventual expansion

Let $I = [0, 1]$, and let $\psi : I \rightarrow I$ be an onto map. A finite interval partition $\mathcal{P} = \{I_1, \dots, I_N\}$ is a *Markov partition* if the restriction of ψ to each atom I_i is a homeomorphism onto a connected union of partition intervals, with endpoint conventions understood up to sets of Lebesgue measure zero. The map is then called Markov. Its incidence matrix $Q = (q_{ij})$ records which intervals are covered by each branch: $q_{ij} = 1$ if the interior of I_j is contained in $\psi(I_i)$, and $q_{ij} = 0$ otherwise. A nonnegative matrix is *primitive* if some positive integer power has all entries strictly positive. These are deterministic interval-map concepts; the term Markov does not mean that the aggregate price process is driven by random shocks. See Boyarsky and Góra (1997, Chapter 9).

A piecewise smooth map is *eventually expanding* if some iterate has derivative

uniformly larger than one in absolute value wherever the derivative exists. As in the interval-map approach used here, we use *observable chaos* to mean eventual expansion. Unlike the existence of a Li–Yorke scrambled set (Li and Yorke, 1975), this condition rules out attracting periodic cycles and, for the primitive piecewise linear Markov maps below, gives an ergodic absolutely continuous invariant probability measure. The explicit densities describe the long-run frequencies of price intervals for almost every initial state.

5.2 Period-5 and higher-period Markov constructions

The simplest construction uses the orbit

$$0 = \varphi^5(0) < \theta_L = \varphi^2(0) < \theta_R = \varphi^4(0) < \varphi(0) < \varphi^3(0) = 1. \quad (36)$$

This gives the partition

$$\mathcal{P}_1 = \{[0, \theta_L), [\theta_L, \theta_R), [\theta_R, \varphi(0)), [\varphi(0), 1]\}. \quad (37)$$

Proposition 2 (Observable chaos on a period-5 Markov partition). *Let $a > 1$ and $0 < b < 1$, and choose*

$$\theta_L = \frac{b}{(1+b)(1+ab)}, \quad \theta_R = \frac{b}{1+b}. \quad (38)$$

Then (35) has the orbit (36) and the Markov partition (37). It is eventually expanding and hence exhibits observable chaos.

Proof. The equations $\varphi_R(\varphi_L(0)) = \theta_L$ and $\varphi_R(1) = \theta_R$ give (38). Together with $\varphi(\theta_L) = 1$ and $\varphi(\theta_R) = 0$, they establish (36). Writing $z = 1/(\theta_R - \theta_L)$, we have

$$z = \frac{(1+b)(1+ab)}{ab^2}.$$

The incidence matrix is

$$Q_{\varphi_5} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (39)$$

Every admissible four-step sequence contains the middle interval. If it also contains the left interval, its slope product has absolute value at least $azb^2 = (1+b)(1+ab) > 1$. If

it does not contain the left interval, it contains at least two middle intervals, and its product is at least $z^2 b^2 > 1$. Therefore

$$\inf |(\varphi^4)'| \geq \min\{(1+b)(1+ab), (zb)^2\} > 1, \quad (40)$$

where the infimum is over points where the derivative exists. $\square$

For higher odd periods, let $S_n := 1 + b + \cdots + b^n$.

Proposition 3 (Observable chaos on a period- $(2n+3)$ Markov partition). *For an integer $n \geq 1$, let $a > 1$, $0 < b < 1$, and*

$$\theta_{2n+3,R} := \frac{b^n}{S_n}, \quad \theta_{2n+3,L} := \frac{b^n}{S_n(1+ab^n)}. \quad (41)$$

Set $\theta_R = \theta_{2n+3,R}$ and $\theta_L = \theta_{2n+3,L}$ in (35). Then the orbit of zero has least period $2n+3$ and defines a Markov partition with $2n+2$ atoms. The map is eventually expanding. More specifically, with

$$z_n := \frac{1}{\theta_R - \theta_L} = \frac{S_n(1+ab^n)}{ab^{2n}},$$

we have

$$\inf |(\varphi^{2n+2})'| \geq \min\{az_nb^{2n}, (z_nb^n)^2\} > 1. \quad (42)$$

Proof. See Appendix C for the ordered orbit, the incidence matrix, and the expansion bound. $\square$

These are dynamical statements about (35). To obtain economies rather than just abstract maps, the corners must also satisfy the parameter mapping in Appendix B. That appendix shows that for every $n \geq 1$ the choice $a = 2$, $b = 1/2$ can be generated by economic parameters satisfying both labor surplus and full capital investment. Thus the higher-period constructions are economically feasible. The Markov equalities define a restricted parameter family; the strict equilibrium inequalities hold in neighborhoods relative to that family.

5.3 Invariant probability densities

An *invariant probability density* for ψ is a nonnegative integrable function p with $P_\psi p = p$ and $\int_I p(x) dx = 1$, where P_ψ is its Frobenius–Perron operator. A nonzero unnormalized

solution is called an invariant eigenfunction. For a piecewise affine Markov map, a density constant on each atom can be obtained from the matrix

$$M_\varphi = (m_{ij}), \quad m_{ij} = \frac{q_{ij}}{|\varphi'_i|}. \quad (43)$$

A nonnegative left eigenvector $\omega M_\varphi = \omega$ gives the atomwise density values; it is normalized by $\sum_i \omega_i |I_i|$. This is the finite Frobenius–Perron method used in Boyarsky and Góra (1997), Matsumoto (2005), and Zhang and Yokoo (2026a,b).

Proposition 4 (Invariant probability density for period 5). *Under Proposition 2, the map has a unique invariant probability density. An unnormalized eigenfunction has values*

$$\omega_5^*(x) = \begin{cases} ab + ab^2, & x \in I_1, \\ ab^2 + ab + 1 + b, & x \in I_2, \\ ab^2, & x \in I_3, \\ ab^2 + b + b^2, & x \in I_4. \end{cases} \quad (44)$$

Its normalized density is $p(x) = \omega_5^*(x) / \sum_{i=1}^4 \omega_i |I_i|$.

Proof. The incidence matrix is (39) and the branch slopes have absolute values (a, z, b, b) . Let $d_1 = \theta_R - \theta_L = ab^2 / [(1+b)(1+ab)]$. Then

$$M_{\varphi_5} = \begin{pmatrix} 0 & 0 & 0 & 1/a \\ d_1 & d_1 & d_1 & d_1 \\ 1/b & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \end{pmatrix}.$$

Substitution verifies $\omega M_{\varphi_5} = \omega$ for (44). Normalization gives an invariant probability density. The incidence matrix is primitive, and Proposition 2 supplies an expanding iterate. The uniqueness result for finite expanding Markov maps therefore gives uniqueness among absolutely continuous invariant probability measures (Boyarsky and Scarowsky, 1979; Boyarsky and Góra, 1997). $\square$

Proposition 5 (Invariant probability density for period $2n+3$). *Under Proposition 3, the map has a unique invariant probability density constant on its $2n+2$ Markov atoms.*

Put $C_n = ab^{2n}$. An unnormalized eigenvector is

$$\begin{aligned}\omega_{2r+1} &= C_n \sum_{h=0}^{n-r} b^{-h}, & r &= 0, \dots, n, \\ \omega_{2r} &= C_n \sum_{h=0}^{n+1-r} b^{-h} + b^{r-1} S_n, & r &= 1, \dots, n+1.\end{aligned}\tag{45}$$

The density is

$$p(x) = \frac{\omega_i}{\sum_{j=1}^{2n+2} \omega_j |I_j|}, \quad x \in I_i.$$

Proof. See Appendix C.1. □

The primitive Markov structure and eventual expansion make this density ergodic. Accordingly, the mass $\int_{I_i} p(x) dx$ is the long-run frequency of visits to I_i for almost every initial price. The statement concerns the absolutely continuous probability measure, not the point masses supported on periodic orbits.

5.4 An explicit economic example

Take

$$\begin{aligned}\beta &= \frac{2}{5}, \quad \pi = \frac{3}{10}, \quad \delta = \frac{1}{10}, \quad D = S = \alpha = 1, \\ A &= \frac{111}{10}, \quad B = \frac{21}{10}, \quad K_L = \frac{13}{12}, \quad K_H = \frac{4}{3}, \quad \xi = \frac{3}{2}.\end{aligned}\tag{46}$$

Then ν_i is uniform on $[3/4, 12/13]$, $a = 2$, $b = 1/2$, and $c = 1/6$. Labor demand never exceeds $13/12 < \xi$, so the wage is always one. The invariant capital interval is $J = [5/6, 7/3]$, and the sufficient return bound is

$$R_t \leq 2 + \frac{5}{7} = \frac{19}{7} < 3 = B + 1 - \delta.$$

Thus full capital investment by investors with an opportunity is strictly optimal throughout J .

The corresponding capital map is

$$H(K) = \begin{cases} 2K + 1/6, & K \leq 13/12, \\ 53/6 - 6K, & 13/12 < K < 4/3, \\ K/2 + 1/6, & K \geq 4/3. \end{cases}\tag{47}$$

Normalization gives $\theta_L = 1/6$, $\theta_R = 1/3$, and

$$\varphi(x) = \begin{cases} 2x + 2/3, & 0 \leq x \leq 1/6, \\ 2 - 6x, & 1/6 < x \leq 1/3, \\ x/2 - 1/6, & 1/3 < x \leq 1. \end{cases} \quad (48)$$

The endpoint orbit is $0 \mapsto 2/3 \mapsto 1/6 \mapsto 1 \mapsto 1/3 \mapsto 0$. The minimum four-step slope product is three, so $\inf |(\varphi^4)'| = 3 > 1$. The normalized invariant density is

$$p(x) = \begin{cases} 9/8, & 0 \leq x < 1/6, \\ 9/4, & 1/6 \leq x < 1/3, \\ 3/8, & 1/3 \leq x < 2/3, \\ 15/16, & 2/3 \leq x \leq 1. \end{cases} \quad (49)$$

Figures 3 and 4 show the map and this density. The example verifies the adoption rule, the wage, saving and portfolio allocation, and eventual expansion in the same economic environment. Figure 5 gives a higher-period example from the economically feasible family in Appendix B.

5.5 Stability under changes in adoption requirements

The closed-form densities above use a particular distribution of project requirements. We now study a specified class of small changes to that distribution. The proof uses a finite Markov partition preserved by the perturbation. Refining this partition so that every branch is affine gives a finite Frobenius–Perron matrix; convergence of its entries and the partition endpoints yields convergence of the densities.

Let $I = [0, 1]$, let λ be Lebesgue measure, and write L^1 for the space of integrable functions with respect to λ . The unperturbed middle branch has the unique fixed point

$$\bar{x} = \frac{\theta_R}{1 + \theta_R - \theta_L}.$$

Fix $\mu > 0$. For sufficiently small $\varepsilon \geq 0$, keep the two outer branches unchanged and

replace the middle branch by

$$\varphi_\varepsilon(x) = \begin{cases} \frac{(1 - \theta_L)\bar{x} - (1 + \mu\varepsilon - \bar{x})x}{\bar{x} - \theta_L(1 + \mu\varepsilon)}, & \theta_L \leq x < \frac{\bar{x}}{1 + \mu\varepsilon}, \\ \frac{-\bar{x}(x - \theta_R)}{(1 + \mu\varepsilon)\theta_R - \bar{x}}, & \frac{\bar{x}}{1 + \mu\varepsilon} \leq x \leq \theta_R. \end{cases} \quad (50)$$

At zero the two lines form φ_M . Their intersection at $c_\varepsilon = \bar{x}/(1 + \mu\varepsilon)$ is a fixed point. The endpoints still map to one and zero, and the middle branch remains continuous and strictly decreasing for small ε . Figure 6 illustrates the change.

This perturbation can be implemented by changing the distribution of project requirements. Return to capital coordinates:

$$H_\varepsilon(K) = \underline{K} + \Delta\varphi_\varepsilon\left(\frac{K - \underline{K}}{\Delta}\right), \quad K \in J,$$

and retain the original outer branches outside J . Define on $[K_L, K_H]$

$$q_\varepsilon(K) = \frac{H_\varepsilon(K) - bK - c}{(a - b)K}. \quad (51)$$

Its endpoint values are one and zero. Since $H'_\varepsilon < 0$ and $H_\varepsilon(K) > c$ on the middle interval,

$$q'_\varepsilon(K) = \frac{KH'_\varepsilon(K) - H_\varepsilon(K) + c}{(a - b)K^2} < 0 \quad (52)$$

where the derivative exists. Therefore

$$F_{\nu,\varepsilon}(v) = q_\varepsilon(1/v), \quad 1/K_H \leq v \leq 1/K_L, \quad (53)$$

extended by zero and one outside this interval, is a valid continuous distribution function. At $\varepsilon = 0$ it is the uniform distribution (26).

Moreover, $[Kq_\varepsilon(K)]' = (H'_\varepsilon(K) - b)/(a - b) < 0$ in the middle interval, so Leontief capital still never exceeds K_L . Thus (29) continues to clear the labor market at α . Since $0 \leq q_\varepsilon \leq 1$, the same bound (25) verifies full capital investment. The perturbations therefore correspond to economies with the same joint-setup technology and asset-market structure, rather than arbitrary changes to the pricing equation.

Proposition 6 (Stability of invariant probability densities). *Let Proposition 3 apply, and define φ_ε by (50). Let $\varepsilon_j \downarrow 0$ and put $\varphi_j = \varphi_{\varepsilon_j}$. For all sufficiently large j , φ_j has a unique invariant probability density p_j , and p_j converges in L^1 to the unique invariant*

probability density p of φ .

Proof. See Appendix D. □

Figure 7 compares the exact invariant densities at $\varepsilon = 0, 0.02, 0.05, 0.10$ and plots their L^1 distances from p_0 . The distances are 0.0152, 0.0370, and 0.0707, respectively: the densities vary with ε , while Proposition 6 establishes their convergence as $\varepsilon \downarrow 0$. The displayed $\varepsilon = 0.10$ is a finite-perturbation illustration; for the period-5 example it also satisfies $\inf |(\varphi_{0.10}^4)'| = 30/17 > 1$.

6 Concluding remarks

We have derived chaotic land-price dynamics from technology choice in an economy with joint setup and heterogeneous adoption requirements. Firms pool finite managerial resources to prepare common operating arrangements. Because providing a given support level becomes more demanding as aggregate capital rises, capital accumulation can make the high-return technology unavailable to an increasing fraction of projects. Entrepreneurs respond optimally by adopting that technology whenever it is feasible and using the linear alternative otherwise.

The mechanism rests on three separate assumptions: preparatory work can be shared, its resource requirement depends on aggregate productive scale, and projects differ in the minimum support they need. The latter two are not consequences of joint provision. Logarithmic preferences and the independence of newly realized project requirements preserve a tractable aggregation, while the labor and asset-market conditions make the derived map consistent with individual decisions.

A uniform distribution of requirements gives a three-branch piecewise linear price map. We connect economic parameters to period- $(2n + 3)$ Markov partitions, verify eventual expansion, and characterize invariant probability densities. A specified family of perturbations of the middle branch can be interpreted as changes in the distribution of requirements; the corresponding invariant densities converge in L^1 .

The analysis takes the joint-setup arrangement as given and studies equilibrium within that institution. Allowing managerial resources to accumulate, introducing a cost of participation, or permitting capital to move between projects after their requirements are known would change the adoption margin and the aggregate dynamics. Persistent heterogeneity could also make the distribution of wealth an additional state variable. These extensions concern the economic scope of the mechanism. The closed-form

dynamical results established here apply to the stated Markov family, rather than to arbitrary adoption distributions or organizational arrangements.

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Declaration of generative AI use

During the preparation and revision of this work, the authors used ChatGPT to assist with mathematical calculations and checks, and manuscript revision. EditGPT was also used for language and readability in preparing earlier versions. The authors are responsible for reviewing and editing the final text and for the content of the article.

Conflict of interest

All four authors declare no conflict of interest.

Data availability

No empirical datasets were used. The figures are generated from the equations and parameter values reported in the paper.

A Saving and equilibrium on an invariant interval

For a price-taking investor, write $s = w - c$ for total saving and let ζ collect the exogenous information and investment opportunities relevant to the individual decision. The gross return to any feasible portfolio does not depend on the investor's own wealth. The continuation value consequently has the form

$$V_t(w, \zeta) = v_t(\zeta) + \frac{1}{1 - \beta} \log w.$$

The terms involving the choice of consumption are

$$\log c + \frac{\beta}{1 - \beta} \log(w - c).$$

Their strictly concave maximization gives $c = (1 - \beta)w$. Portfolio returns affect v_t but not this propensity. On the compact paths considered here, positive asset returns are bounded above and away from zero, so the discounted continuation terms are well defined. This argument applies before the next project's requirement is learned and allows its return to be uncertain.

Proof of Lemma 1. Fix $K_0 \in J$ and generate $K_{t+1} = H(K_t)$. Invariance keeps the path in J . Set

$$P_t = \frac{1 - \pi}{\pi S} K_{t+1}, \quad W_t = \frac{K_{t+1}}{\beta \pi}.$$

By the definition of H and d_0 ,

$$W_t = (B + 1 - \delta)K_t + (A - B)K_t F_\nu(1/K_t) + (P_t + D)S.$$

Choose the full-participation setup arrangement. Proposition 1 gives $u_t = 1/K_t$, wage α , optimal technology choices, and labor-market clearing at every date.

Equation (24) and (25) imply $R_t < B + 1 - \delta$ for every t . The minimum payoff on any feasible productive investment therefore exceeds the payoff on land. For investors with an opportunity, full investment in capital is strictly optimal; the only asset available to the remaining investors is land. The logarithmic saving rule then prescribes consumption $(1 - \beta)w_{it}$ and saving βw_{it} for each investor.

Starting with the given asset holdings, define subsequent holdings using these individual policies. By the wealth-weighted aggregation assumption in Section 2.3, these policies generate aggregate capital demand $\beta \pi W_t = K_{t+1}$ and land demand

$\beta(1 - \pi)W_t/P_t = S$. Applying the capital-weighted aggregation assumption to the next period's project requirements gives aggregate Leontief capital $K_{t+1}F_\nu(1/K_{t+1})$. Induction therefore reproduces the proposed aggregate path and wealth equation from individual budgets. Positive initial wealth and positive returns ensure positive consumption.

Finally, add the individual budgets and cancel land purchases against the value of the existing land stock. With the workers' consumption included, the goods-market identity is

$$C_t + C_t^W + K_{t+1} = Y_t + (1 - \delta)K_t + DS.$$

Thus goods, labor, and land markets clear. Individual decisions use the same aggregate price sequence as the one they generate, establishing the asserted equilibrium. $\square$

B Economic implementation of the normalized parameters

The normalized corners must correspond to positive economic thresholds and to a positive intercept in the capital map. The following construction replaces a purely reduced-form choice of parameters.

Lemma 2 (From the price map to economic parameters). *Fix $a > 1$, $0 < b < 1$, and $0 < \theta_L < \theta_R < 1$. Define*

$$\ell_* := \frac{1 - a\theta_L + b\theta_R}{a - b}, \quad c_* := (1 - b)\ell_* - b\theta_R. \quad (54)$$

Suppose $\ell_* > 0$ and $c_* > 0$. Then these parameters can be generated by an economy satisfying (26), (29), (33), and (25).

Proof. Choose $\beta \in (0, 1)$, $\delta \in [0, 1]$, and $D, S > 0$. For a sufficiently small $\pi \in (0, 1)$, put

$$d_0 = 1 - \beta(1 - \pi), \quad A = \frac{ad_0}{\beta\pi} - 1 + \delta, \quad B = \frac{bd_0}{\beta\pi} - 1 + \delta, \quad c = \frac{\beta\pi DS}{d_0}.$$

Then $A > B > 0$. Let

$$\Delta = \frac{c}{c_*}, \quad \underline{K} = \Delta\ell_*, \quad K_L = \Delta(\ell_* + \theta_L), \quad K_H = \Delta(\ell_* + \theta_R), \quad \overline{K} = \Delta(\ell_* + 1). \quad (55)$$

The first definition in (54) implies $(a - b)\ell_* + a\theta_L - b\theta_R = 1$, and the second implies

$c = (1 - b)\underline{K} - b\Delta\theta_R$. It follows that $bK_H + c = \underline{K}$ and $aK_L + c = \overline{K}$. The order of the corners and $\ell_* > 0$ give (33). Choose any $\xi > K_L$ and $\alpha > 0$, and let the requirements have distribution (26). This verifies the setup, labor, and map-normalization conditions.

Using (55), the sufficient asset-return condition becomes

$$a + \frac{c_*}{\beta(1 - \pi)\ell_*} < \frac{bd_0}{\beta\pi}. \quad (56)$$

As $\pi \downarrow 0$, the left-hand side has a finite limit and the right-hand side diverges. Hence the inequality holds for sufficiently small π , completing the construction. $\square$

There are economically feasible members of the Markov family for every $n \geq 1$. To see this explicitly, take $a = 2$, $b = 1/2$, and the corners in (41). With $t = 2^{-n} \leq 1/2$,

$$\theta_R = \frac{t}{2 - t}, \quad \theta_L = \frac{t}{(2 - t)(1 + 2t)}.$$

These satisfy $2\theta_L + \theta_R < 1$, because the difference between the denominator $(2 - t)(1 + 2t)$ and the numerator $t(3 + 2t)$ is $2 - 4t^2 > 0$. Therefore

$$\ell_* > \theta_R > 0, \quad c_* = \frac{1}{2}(\ell_* - \theta_R) > 0.$$

Lemma 2 applies. One common choice of asset-market parameters for all these constructions is

$$\beta = \frac{2}{5}, \quad \pi = \frac{1}{10}, \quad \delta = \frac{1}{10}, \quad D = S = \alpha = 1, \quad A = \frac{311}{10}, \quad B = \frac{71}{10}. \quad (57)$$

Then $d_0 = 16/25$, $c = 1/16$, and the worst capital gross return is eight. Since $c_*/\ell_* < 1/2$, the left-hand side of (56) is below

$$2 + \frac{1/2}{(2/5)(9/10)} = \frac{61}{18} < 8.$$

Use (55) for each n and select $\xi > K_L$. In fact, $\xi = 2$ works for the entire family: the inequalities $\ell_* - \theta_R \geq 2/9$ and $\ell_* + \theta_L \leq 5/6$ imply $K_L \leq 15/32 < 2$. The period-5 example in (46) uses a different, equally feasible choice of π to give especially simple capital thresholds.

The construction establishes nonemptiness, not an open set satisfying the Markov equalities in the full unrestricted parameter space. The equilibrium inequalities are

strict, so they continue to hold for small changes within the parameterized Markov family.

C The higher-period Markov construction

Proof of Proposition 3. Write $T(x) = b(x - \theta_R)$ for the right affine branch and $v_0 = 1 - a\theta_L$. Solving

$$T^n(1) = \theta_R, \quad T^n(v_0) = \theta_L$$

gives (41). Put $w_j = T^j(1)$ and $v_j = T^j(v_0)$. We have

$$w_n = \theta_R, \quad v_n = \theta_L, \quad \varphi(\theta_L) = 1, \quad \varphi(\theta_R) = 0.$$

Furthermore,

$$v_0 - w_1 = \frac{1}{S_n(1 + ab^n)} > 0, \quad w_0 - v_0 = a\theta_L > 0.$$

Since T is strictly increasing, its iterates preserve these inequalities. Thus the ordered orbit endpoints are

$$0 < \theta_L < \theta_R < v_{n-1} < w_{n-1} < \cdots < v_0 < w_0 = 1, \quad (58)$$

where for $n = 1$ the intermediate chain is simply $v_0 < 1$. Along the actual orbit,

$$0 \mapsto v_0 \mapsto v_1 \mapsto \cdots \mapsto v_n = \theta_L \mapsto 1 \mapsto w_1 \mapsto \cdots \mapsto w_n = \theta_R \mapsto 0.$$

All $2n + 3$ orbit points before the final return are distinct, so the least period is $2n + 3$.

Let $N = 2n + 2$ and enumerate the intervals between the endpoints in (58) from left to right. The first has slope a , the second slope $-z_n$, and the remaining $2n$ intervals slope b . Their incidence matrix satisfies

$$q_{1N} = 1, \quad q_{2j} = 1 \quad (1 \leq j \leq N), \quad q_{j,j-2} = 1 \quad (3 \leq j \leq N), \quad (59)$$

with all other entries zero. In particular, every atom reaches the middle atom, the middle atom reaches every atom, and it has a self-loop. The matrix is primitive.

Every admissible sequence of N successive intervals contains the middle atom. Otherwise a path can move only down the odd-indexed chain, through the left atom,

and down the even-indexed chain; it reaches the middle within $N - 1$ transitions. If the sequence contains both the left and middle atoms, its derivative product is at least az_nb^{2n} , since all other slope magnitudes are at least b .

If it does not contain the left atom, every run of right-branch atoms has length at most n . A sequence of length $2n + 2$ must then contain at least two middle atoms, and its derivative product is at least $z_n^2b^{2n}$. Finally,

$$az_nb^{2n} = S_n(1 + ab^n) > 1, \quad z_nb^n = S_n\left(1 + \frac{1}{ab^n}\right) > 1.$$

These two cases establish (42). □

C.1 The invariant eigenvector

Proof of Proposition 5. The incidence matrix (59) gives

$$M_{\varphi_{2n+3}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1/a \\ d_n & d_n & \cdots & d_n & d_n & d_n \\ 1/b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1/b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/b & 0 & 0 \end{pmatrix}, \quad d_n = \frac{ab^{2n}}{S_n(1 + ab^n)}.$$

Set $C_n = ab^{2n}$. The eigenvector equations, with the normalization $d_n\omega_2 = C_n$, are

$$\omega_j = C_n + \frac{\omega_{j+2}}{b} \quad (1 \leq j \leq 2n), \quad \omega_{2n+1} = C_n, \quad \omega_{2n+2} = C_n + \frac{\omega_1}{a}.$$

Iterating these relations gives (45). In particular,

$$\omega_1 = ab^n S_n, \quad \omega_2 = (1 + ab^n)S_n,$$

which verifies $d_n\omega_2 = C_n$ and hence all the equations. All entries are positive. Normalizing by the interval lengths gives an invariant probability density. Primitivity and the expanding iterate in Proposition 3 imply its uniqueness and ergodicity, by the same finite Markov-map result used for Proposition 4. □

D Stability of invariant probability densities

We use the standard uniqueness result for piecewise C^2 expanding Markov interval maps (Boyarsky and Scarowsky, 1979; Boyarsky and Góra, 1997): a finite Markov map with a primitive incidence matrix and an expanding iterate has at most one absolutely continuous invariant probability measure. The piecewise affine maps considered here satisfy the smoothness requirements on their affine partition atoms. We construct their densities directly below, so only the uniqueness part of this result is needed.

Proof of Proposition 6. Fix $n \geq 1$ and the period- $(2n+3)$ Markov case in Proposition 3. Let $\mathcal{P}_n$ denote its Markov partition. By Proposition 3, φ is eventually expanding. Hence there exist an integer $m \geq 1$ and a number $\rho > 1$ such that

$$\inf \{ |(\varphi^m)'(x)| : (\varphi^m)'(x) \text{ exists} \} \geq \rho > 1. \quad (60)$$

We first show that the same finite Markov partition and incidence matrix, together with a common expanding iterate, persist for all sufficiently small perturbations.

Let

$$c_\varepsilon := \frac{\bar{x}}{1 + \mu\varepsilon}, \quad c_0 = \bar{x}.$$

Direct substitution into (50) gives

$$\varphi_\varepsilon(\theta_L) = 1, \quad \varphi_\varepsilon(c_\varepsilon) = c_\varepsilon, \quad \varphi_\varepsilon(\theta_R) = 0. \quad (61)$$

For sufficiently small ε , $\theta_L < c_\varepsilon < \theta_R$. Moreover, φ_ε is continuous and strictly decreasing on $[\theta_L, \theta_R]$, with image $[0, 1]$, and the outer branches coincide with those of φ . The period- $(2n+3)$ orbit used to define $\mathcal{P}_n$ is therefore unchanged, and $\mathcal{P}_n$ remains a Markov partition for φ_ε with the same incidence matrix as for φ .

The incidence matrix of $\mathcal{P}_n$ is primitive. Indeed, every atom reaches the middle atom $I_c = [\theta_L, \theta_R]$ after finitely many iterations, while I_c maps onto the whole interval and therefore reaches every atom. Moreover, $I_c \subset \varphi_\varepsilon(I_c) = [0, 1]$, so a path can remain at I_c for an arbitrary number of additional steps. Since $\mathcal{P}_n$ has finitely many atoms, there exists a common integer $k \geq 1$ such that a path of length k exists from every atom to every other atom. Hence, if Q_n denotes the incidence matrix of $\mathcal{P}_n$, then $Q_n^k > 0$ entrywise, and Q_n is primitive. This argument holds uniformly for all sufficiently small ε .

To control expansion, refine $\mathcal{P}_n$ by the point c_ε and denote the resulting partition by

$$\widehat{\mathcal{P}}_{n,\varepsilon} = \{I_1(\varepsilon), \dots, I_N(\varepsilon)\}.$$

On every atom of $\widehat{\mathcal{P}}_{n,\varepsilon}$, the map φ_ε is affine and one-to-one. In particular,

$$\varphi_\varepsilon([\theta_L, c_\varepsilon]) = [c_\varepsilon, 1], \quad \varphi_\varepsilon([c_\varepsilon, \theta_R]) = [0, c_\varepsilon]. \quad (62)$$

Consequently, the incidence pattern of $\widehat{\mathcal{P}}_{n,\varepsilon}$ is independent of ε for all sufficiently small ε .

This refined incidence matrix is also primitive. The two middle subatoms reach each other, the lower subatom reaches all the right-branch atoms, and the upper subatom reaches the left atom. Together with the outer transitions in (59), this gives irreducibility. There is a two-step cycle between the middle subatoms and an odd cycle of length $2n + 3$: from the lower subatom take the odd-indexed right chain to the left atom, then the even-indexed chain to the upper subatom, and return to the lower subatom. Thus the period of the refined graph is one.

The slopes of the two perturbed middle subbranches are

$$s_-(\varepsilon) = -\frac{1 + \mu\varepsilon - \bar{x}}{\bar{x} - \theta_L(1 + \mu\varepsilon)}, \quad s_+(\varepsilon) = -\frac{\bar{x}}{(1 + \mu\varepsilon)\theta_R - \bar{x}}. \quad (63)$$

They are nonzero and continuous in ε , and at $\varepsilon = 0$ both coincide with the slope of the unperturbed middle branch. The other slopes are a and b and do not depend on ε .

Let $Q = (q_{ij})$ denote the incidence matrix of $\widehat{\mathcal{P}}_{n,\varepsilon}$, which is independent of ε for all sufficiently small ε . Define

$$\mathcal{A}_m := \{(i_0, \dots, i_{m-1}) : q_{i_r, i_{r+1}} = 1, \quad r = 0, \dots, m-2\}.$$

We refer to the elements of $\mathcal{A}_m$ as *admissible m -step itineraries*. The set $\mathcal{A}_m$ is finite and independent of ε . For $\mathbf{i} \in \mathcal{A}_m$, define the corresponding slope product by

$$S_{\mathbf{i}}(\varepsilon) := \prod_{r=0}^{m-1} s_{i_r}(\varepsilon),$$

where each $s_{i_r}(\varepsilon)$ is one of a , b , $s_-(\varepsilon)$, and $s_+(\varepsilon)$. Whenever $(\varphi_\varepsilon^m)'(x)$ exists, its value is one of the products $S_{\mathbf{i}}(\varepsilon)$.

At $\varepsilon = 0$, (60) implies

$$|S_{\mathbf{i}}(0)| \geq \rho > 1 \quad \text{for every } \mathbf{i} \in \mathcal{A}_m.$$

Each $S_{\mathbf{i}}(\varepsilon)$ is continuous in ε , and $\mathcal{A}_m$ is finite. Hence there exist $\varepsilon_0 > 0$ and $\rho_0 > 1$ such that

$$\min_{\mathbf{i} \in \mathcal{A}_m} |S_{\mathbf{i}}(\varepsilon)| \geq \rho_0 > 1, \quad 0 \leq \varepsilon < \varepsilon_0. \quad (64)$$

Equivalently,

$$\inf \{ |(\varphi_\varepsilon^m)'(x)| : (\varphi_\varepsilon^m)'(x) \text{ exists} \} \geq \rho_0 > 1.$$

Thus every sufficiently small perturbation is eventually expanding, with the same iterate m .

Moreover, φ_ε is a finite piecewise affine Markov interval map on $\widehat{\mathcal{P}}_{n,\varepsilon}$, and the incidence matrix of this affine partition is primitive, as shown above. Therefore, the standard uniqueness result stated above, together with (64), implies that φ_ε admits at most one absolutely continuous invariant probability measure, or equivalently, at most one invariant probability density, for every $0 \leq \varepsilon < \varepsilon_0$.

We now construct that invariant probability density directly. For the refined partition $\widehat{\mathcal{P}}_{n,\varepsilon}$, let

$$\ell_i(\varepsilon) := \lambda(I_i(\varepsilon)), \quad \ell_\varepsilon := (\ell_1(\varepsilon), \dots, \ell_N(\varepsilon))^\top,$$

and let $s_i(\varepsilon)$ denote the slope of φ_ε on $I_i(\varepsilon)$. Using the same incidence matrix $Q = (q_{ij})$, define

$$M_\varepsilon = (m_{ij}(\varepsilon)), \quad m_{ij}(\varepsilon) := \frac{q_{ij}}{|s_i(\varepsilon)|}. \quad (65)$$

Since φ_ε maps $I_i(\varepsilon)$ affinely onto the union of those $I_j(\varepsilon)$ for which $q_{ij} = 1$,

$$|s_i(\varepsilon)|\ell_i(\varepsilon) = \sum_{j=1}^N q_{ij}\ell_j(\varepsilon).$$

Therefore

$$M_\varepsilon \ell_\varepsilon = \ell_\varepsilon. \quad (66)$$

If $D_\varepsilon := \text{diag}(\ell_1(\varepsilon), \dots, \ell_N(\varepsilon))$, then (66) implies that $D_\varepsilon^{-1}M_\varepsilon D_\varepsilon$ is row-stochastic. Hence the spectral radius of M_ε is one. By the Perron–Frobenius theorem, there exists a nonnegative left eigenvector $v_\varepsilon = (v_{\varepsilon,1}, \dots, v_{\varepsilon,N})$ satisfying

$$v_\varepsilon M_\varepsilon = v_\varepsilon. \quad (67)$$

Normalize it by

$$v_\varepsilon \ell_\varepsilon = \sum_{i=1}^N v_{\varepsilon,i} \ell_i(\varepsilon) = 1. \quad (68)$$

Define

$$p_\varepsilon(x) := \sum_{i=1}^N v_{\varepsilon,i} \mathbf{1}_{I_i(\varepsilon)}(x), \quad (69)$$

where $\mathbf{1}_A$ denotes the indicator function of a set A . For x in the interior of $I_j(\varepsilon)$, each atom $I_i(\varepsilon)$ with $q_{ij} = 1$ contains exactly one preimage of x . Hence the Frobenius–Perron operator P_ε of φ_ε satisfies

$$\begin{aligned} (P_\varepsilon p_\varepsilon)(x) &= \sum_{i:q_{ij}=1} \frac{v_{\varepsilon,i}}{|s_i(\varepsilon)|} \\ &= (v_\varepsilon M_\varepsilon)_j = v_{\varepsilon,j} = p_\varepsilon(x). \end{aligned}$$

Thus p_ε is an invariant probability density. By the uniqueness established above, it is the unique invariant probability density of φ_ε for all sufficiently small ε .

It remains to prove convergence. The endpoints of the refined partition depend continuously on ε : the endpoints inherited from $\mathcal{P}_n$ are fixed, while $c_\varepsilon \rightarrow c_0$. Therefore

$$\ell_\varepsilon \longrightarrow \ell_0. \quad (70)$$

The zero pattern Q is fixed, and all branch slopes are continuous in ε by (63), so

$$M_\varepsilon \longrightarrow M_0 \quad (71)$$

entrywise. Since every atom of $\widehat{\mathcal{P}}_{n,0}$ has positive length and there are finitely many atoms, there exist $\delta > 0$ and $\varepsilon_1 > 0$ such that

$$\ell_i(\varepsilon) \geq \delta$$

for every i and every $0 \leq \varepsilon < \varepsilon_1$. From the normalization (68),

$$0 \leq v_{\varepsilon,i} \leq \frac{1}{\delta},$$

so the family $\{v_\varepsilon\}$ is bounded in the finite-dimensional space $\mathbb{R}^N$.

Take any sequence $\varepsilon_k \downarrow 0$. It has a subsequence, again denoted ε_k , such that $v_{\varepsilon_k} \rightarrow v^*$.

Passing to the limit in (67) and (68) by (70) and (71) gives

$$v^* M_0 = v^*, \quad v^* \ell_0 = 1, \quad v^* \geq 0.$$

Accordingly,

$$p^*(x) := \sum_{i=1}^N v_i^* \mathbf{1}_{I_i(0)}(x)$$

is an invariant probability density of φ . The preceding uniqueness argument, applied at $\varepsilon = 0$, implies that $p^* = p_0$. Since all atoms $I_i(0)$ have positive length, equality of these piecewise constant functions implies $v^* = v_0$. Every convergent subsequence therefore has the same limit, and hence

$$v_\varepsilon \longrightarrow v_0. \tag{72}$$

Finally, corresponding partition intervals satisfy

$$\lambda(I_i(\varepsilon) \triangle I_i(0)) \longrightarrow 0$$

for every i , where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of two sets. Using (69) and adding and subtracting $\sum_i v_{\varepsilon,i} \mathbf{1}_{I_i(0)}$, we obtain

$$\begin{aligned} \|p_\varepsilon - p_0\|_1 &\leq \sum_{i=1}^N v_{\varepsilon,i} \lambda(I_i(\varepsilon) \triangle I_i(0)) \\ &\quad + \sum_{i=1}^N |v_{\varepsilon,i} - v_{0,i}| \lambda(I_i(0)). \end{aligned}$$

The first term tends to zero because the $v_{\varepsilon,i}$ are uniformly bounded, and the second tends to zero by (72). Thus

$$\|p_\varepsilon - p_0\|_1 \longrightarrow 0.$$

At $\varepsilon = 0$, $\widehat{\mathcal{P}}_{n,0}$ is only a refinement of the Markov partition used in the preceding invariant-density proposition. Hence p_0 and the normalized density p stated there are both invariant probability densities of φ ; by uniqueness, $p_0 = p$. Taking $\varepsilon = \varepsilon_j$ and writing $p_j := p_{\varepsilon_j}$ proves

$$\|p_j - p\|_1 \longrightarrow 0.$$

This proves Proposition 6. □

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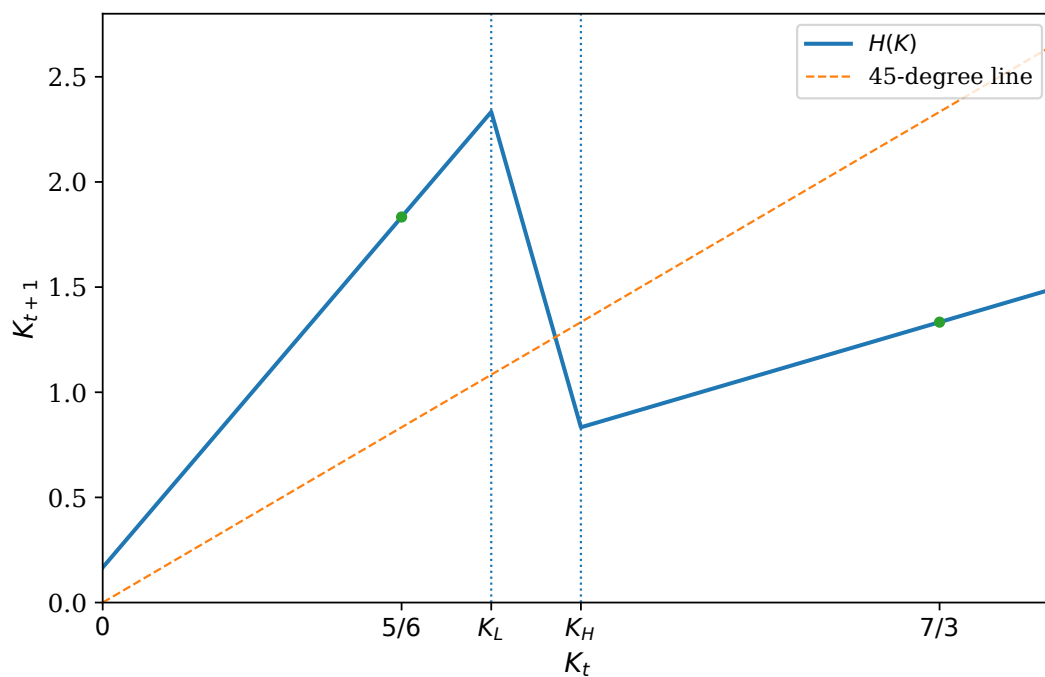


Figure 1: Capital dynamics in the economy of (46). The solid graph is (47), the dashed line is the 45-degree line, and the dotted lines identify the adoption thresholds $K_L = 13/12$ and $K_H = 4/3$. The invariant interval is $[5/6, 7/3]$.

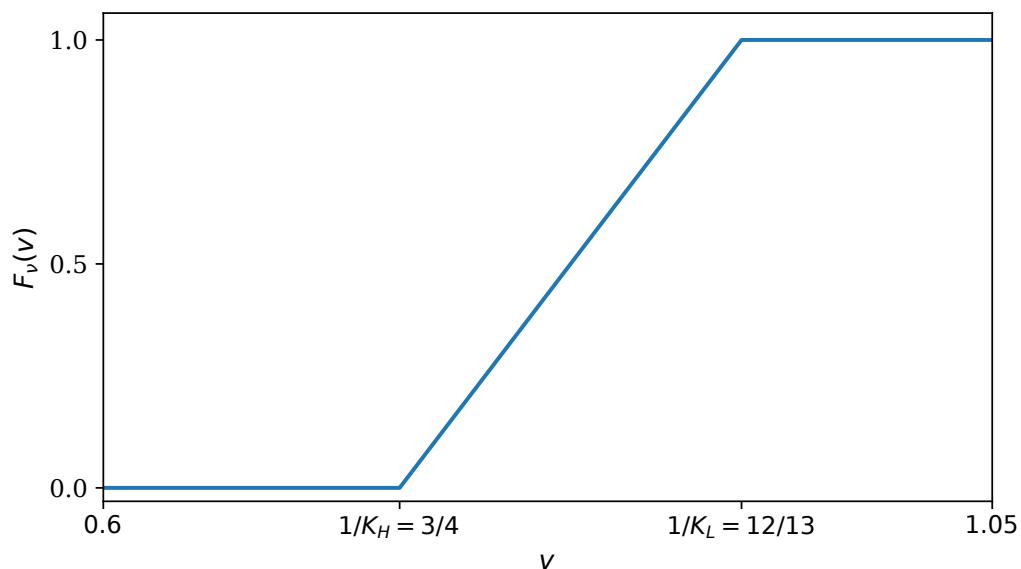


Figure 2: The cumulative distribution F_v of actual project requirements in (46): a uniform distribution on $[3/4, 12/13]$. The argument v denotes the support level required by a project.

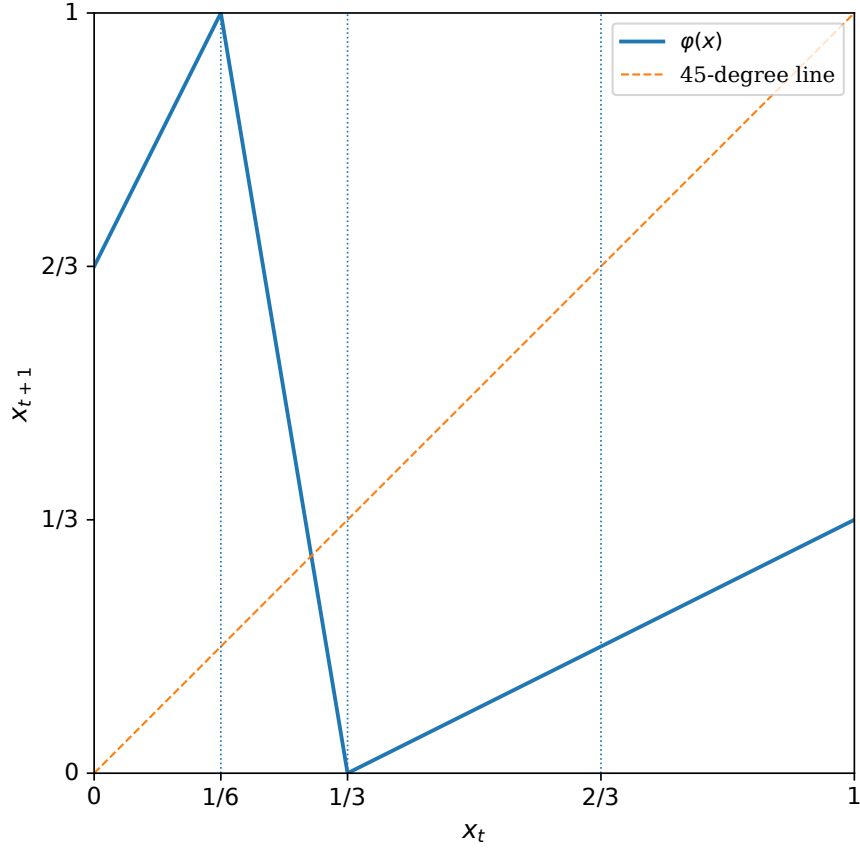


Figure 3: The period-5 Markov map (48), with $a = 2$, $b = 1/2$, $\theta_L = 1/6$, and $\theta_R = 1/3$. The endpoints follow $0 \mapsto 2/3 \mapsto 1/6 \mapsto 1 \mapsto 1/3 \mapsto 0$.

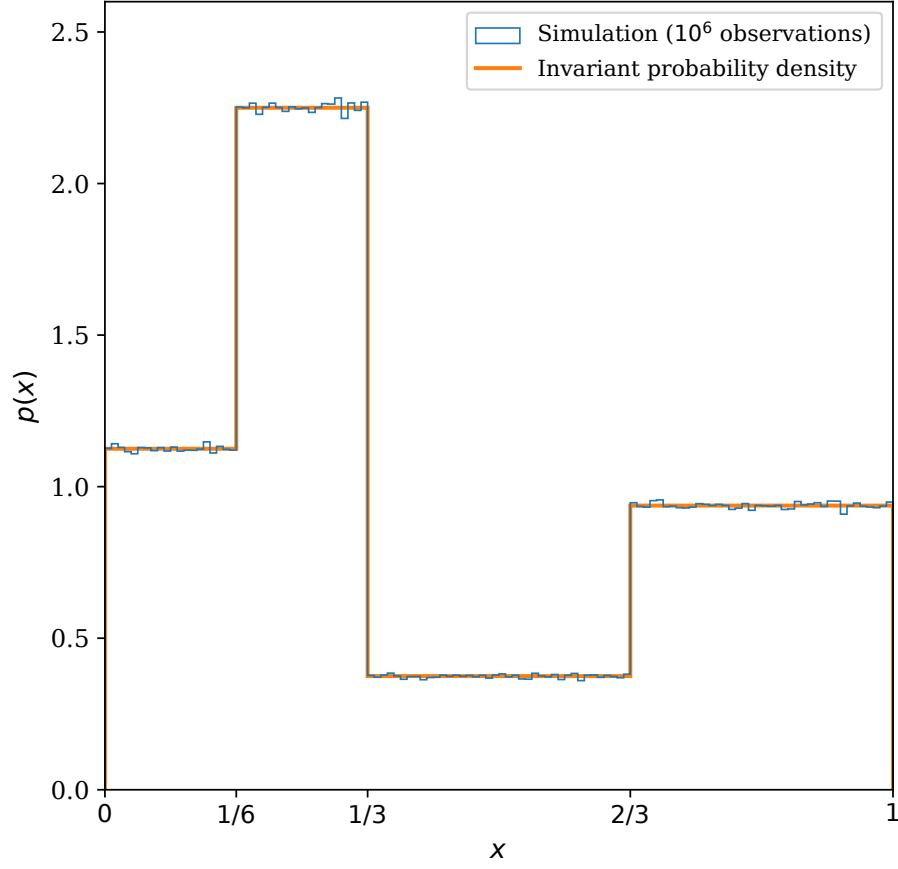


Figure 4: The invariant probability density (49) and a simulated histogram for (48). The histogram uses 10^6 post-transient observations, after discarding 10^4 iterations from $x_0 = 0.123456789123$. The theoretical density is normalized to unit integral.

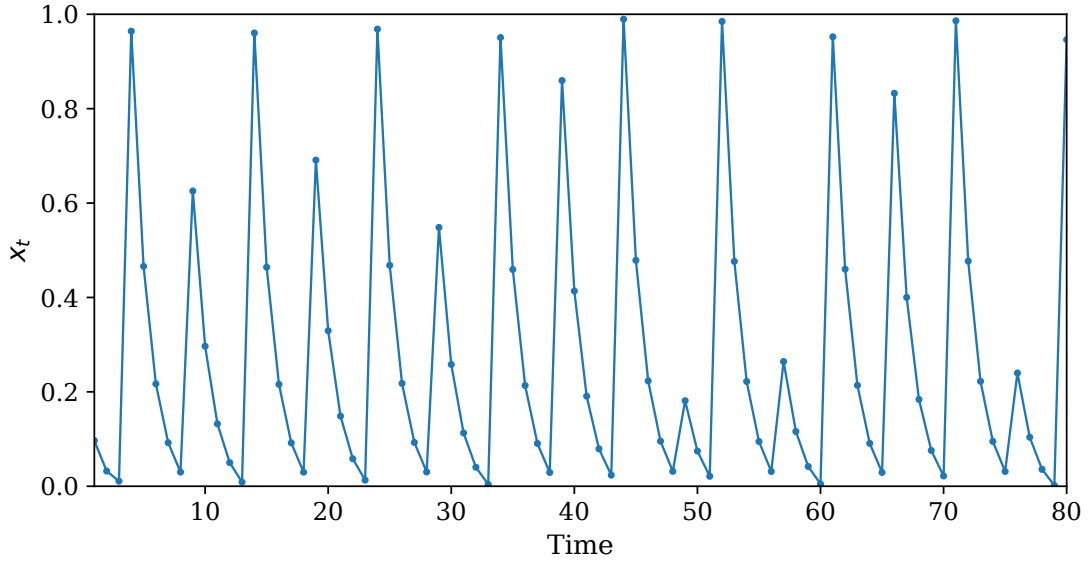


Figure 5: An endogenous price trajectory for the period-11 Markov family ($n = 4$, $a = 2$, $b = 1/2$). The corners are given by (41). Economic parameters are (57), with thresholds determined by (55) and $\xi = 2$. The plot reports 80 observations after 10^4 discarded iterations from $x_0 = 0.123456789123$.

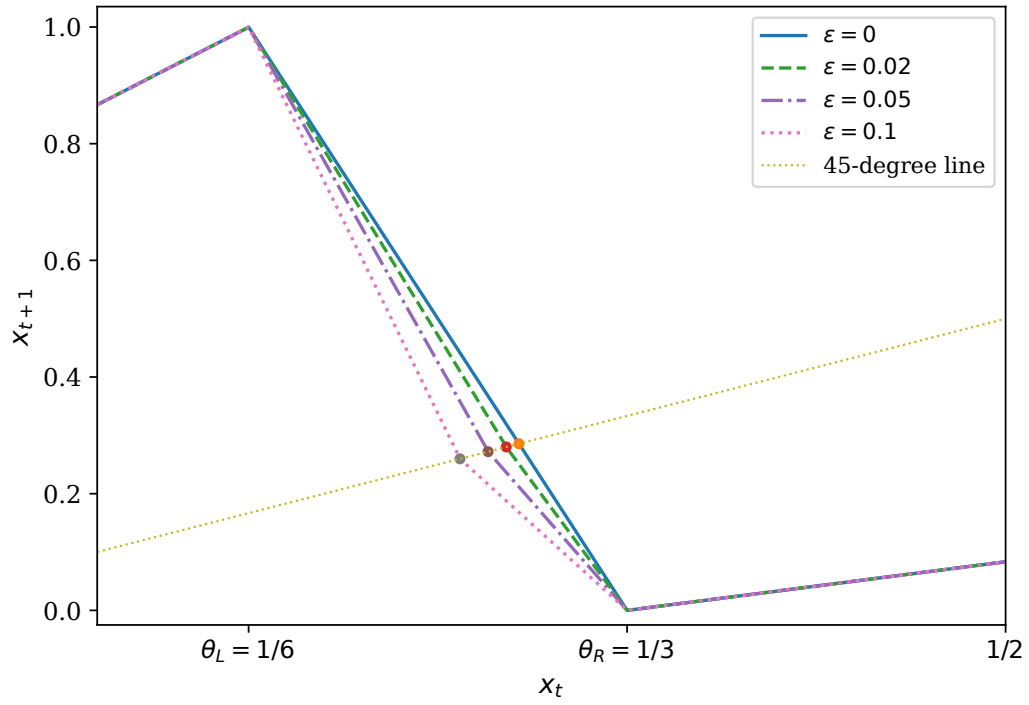


Figure 6: Changes in the middle branch under (50), with the period-5 parameters of (48), $\mu = 1$, and $\varepsilon \in \{0, 0.02, 0.05, 0.10\}$. The outer branches remain unchanged. Each displayed map is implemented by the requirement distribution in (53).

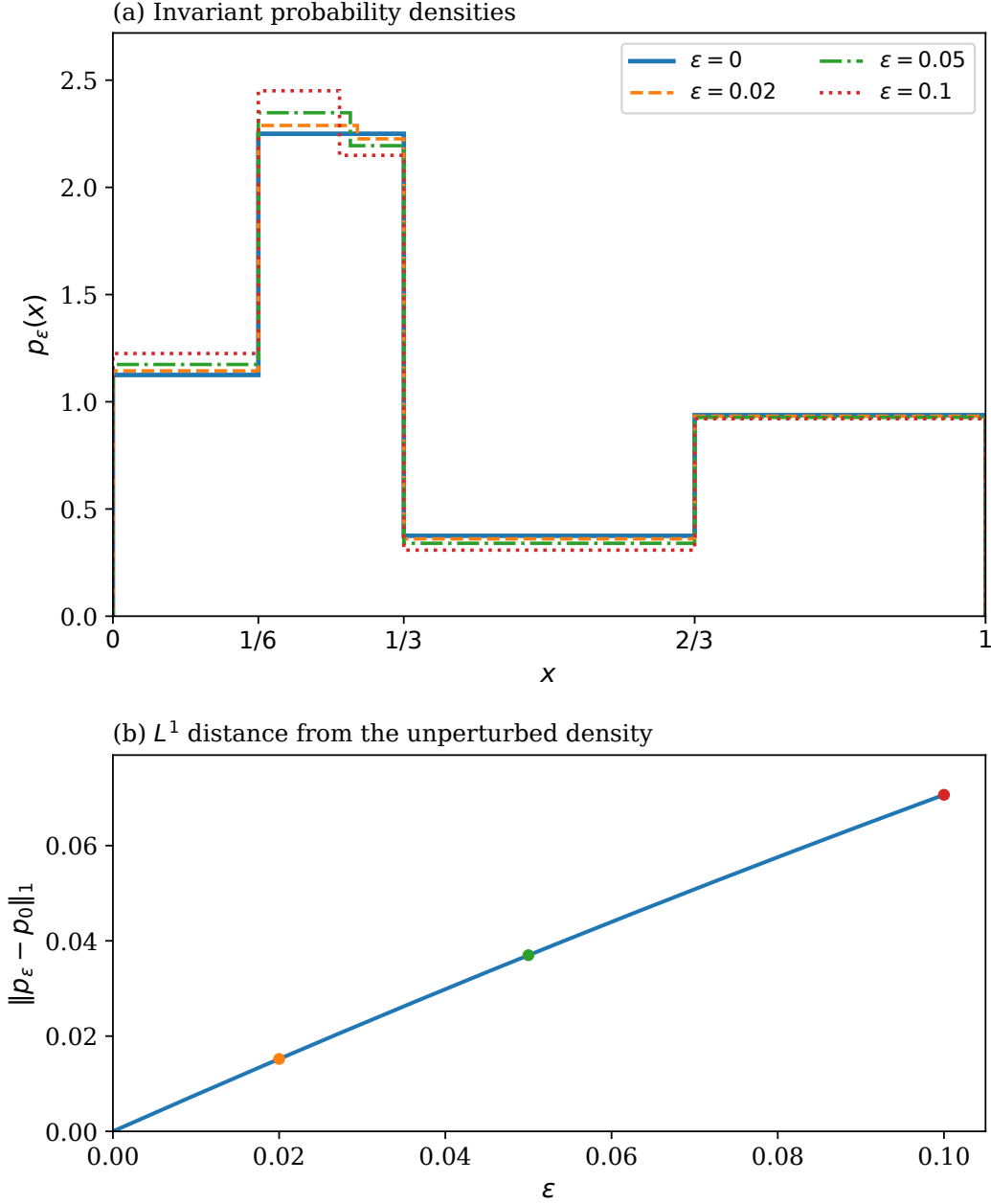


Figure 7: How the invariant probability density changes under (50), for $a = 2$, $b = 1/2$, $\theta_L = 1/6$, $\theta_R = 1/3$, and $\mu = 1$. Panel (a) compares the exact densities p_ε at $\varepsilon = 0, 0.02, 0.05, 0.10$; panel (b) plots the exact L^1 distance $\|p_\varepsilon - p_0\|_1$. Densities are calculated from the finite Frobenius–Perron matrices on the refined Markov partitions, not from orbit simulations. Proposition 6 is a local result as $\varepsilon \downarrow 0$; $\varepsilon = 0.10$ is also checked directly for this period-5 example.