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Abstract

This paper identifies the conditions under which an overlapping-generations model incorporating imperfect technology choice yields observable chaotic behavior and analytically characterizes an ergodic Markov chain describing the dynamics. Furthermore, we demonstrate that this chaotic behavior exhibits some robustness to deterministic perturbations. These findings are useful not only for detecting chaotic behavior in economic models but also for predicting long-term chaotic economic fluctuations.

JEL classification: C63, E32, O41.
Keywords: Heterogeneity; Chaotic dynamics; Overlapping generations model; Piecewise linearity; Ergodic Markov chain

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1 Introduction

Over the past several decades, the real economy has frequently experienced highly irregular and unpredictable fluctuations. Perspectives on the causes of these economic fluctuations vary. Some researchers argue that the observed economic instability stems from external shocks and their propagation. Others, employing nonlinear models to overcome the limitations of linear stochastic models, seek to explain these complex patterns through the inherent dynamics of the economy itself.

The theoretical foundations for complex behavior, particularly how chaotic dynamics emerge from simple economic structures, were extensively developed in the 1980s. For instance, it traces back to pioneering studies by Day (1982), Benhabib and Day (1982), Grandmont (1985), Deneckere and Pelikan (1986), and Boldrin and Montrucchio (1986). Among these, research on chaotic dynamics based on overlapping generations (OLG) models continues to this day. Studies such as Chen et al. (2008), Fanti and Gori (2011), and Asano et al. (2024) attempt to extend the OLG model to various contexts and capture self-sustaining chaotic dynamics. These findings reveal the role that nonlinearity plays in autonomous deterministic fluctuations without external shocks. Recent studies such as Deng et al. (2022), Hommes et al. (2024), and Alexeeva et al. (2025) also examine chaotic dynamics within the OLG framework.

This paper is motivated by the need to identify the underlying mechanisms of endogenous fluctuations using nonlinear general equilibrium models. However, its primary objective is to develop a novel and effective approach to understanding such recursive yet nonperiodic economic fluctuations. This is because existing research has primarily focused on detecting chaotic behavior, which alone cannot fully describe its characteristics. To bridge this gap, this paper contributes to the literature by revealing chaotic economic fluctuation patterns through their association with ergodic Markov chains.

This paper constructs a continuous piecewise linear OLG model exhibiting chaotic fluctuations based on the model proposed by Umezuki and Yokoo (2019). They demonstrate that by incorporating a structure in which firm owners choose between two technologies to maximize capital income each period into the standard Diamond-type OLG framework, cyclical patterns are driven by endogenous thresholds. This technology choice is also adopted by Gori et al. (2026). Alternative choices can be found in, for example, Marsiglio and Tolotti (2018), and Zeppini (2015).

However, the technology choice in Umezuki and Yokoo (2019) relies on the simplified assumption that the threshold is perfectly known to economic agents. We relax this assumption by incorporating information imperfections in the real world and observation errors due to agent heterogeneity, thereby examining the impact of imperfect technology choice on economic fluctuations.

Interestingly, it is precisely this consideration that gives rise to the structure that randomizes the economy around the threshold and directly accounts for the emergence of observable chaos. Notably, such long-run chaotic behavior makes it possible to reproduce the recurrent but not periodic fluctuations observed in business cycles, thereby overcoming the limitation of Umezuki and Yokoo(2019), which is unable to explain such dynamic patterns.

To conduct a detailed analysis of long-term chaotic dynamics, this paper characterizes the structure of ergodic Markov chains in the OLG model. When the economic system in our model exhibits the Markov property, the economic state space is partitioned into finite intervals, and the transition patterns between these intervals follow ergodic Markov chains. This allows us to derive a degree of predictable long-term behavior in chaotic economic systems, including invariant measures and transition patterns. This result is somewhat paradoxical and intriguing. This is because, although chaotic behavior is deterministic (meaning future economic states depend on initial values), the Markov property implies that the future state depends only on the present state.

Furthermore, to generalize our results, we prove that the invariant measure remains stable under small deterministic perturbations. This ensures that the stationary distribution of chaotic behavior maintains a certain robustness. In other words, although the distribution of observation errors satisfying piecewise linearity is highly specific, we demonstrate that several important properties are preserved even when this restrictive functional form is relaxed. Specifically, we show that observable chaos persists even under small perturbations and that its stationary distribution (or invariant measure) is “close” to the stationary distribution of the chaotic behavior when unperturbed, which can be explicitly derived.

This paper offers an effective approach to detecting, describing, and predicting chaotic fluctuations. The techniques employed to compute invariant measures are partly borrowed from mathematical tools such as a matrix version of the Frobenius-Perron operator. See Boyarsky and Góra (1997) for mathematical details. See also, for instance, Huang and Day (1993), Matsumoto (2005), Yokoo and Ishida (2008), and Asano and Yokoo (2019) for their applications in different economic settings.

This paper is organized as follows. Section 2 presents the model and describes the imperfect technology choice. Section 3 captures the deterministic chaotic patterns using the Markov property. Section 4 derives the main results on the emergence of the ergodic Markov chain. Section 5 discusses the robustness of invariant measures. Section 6 provides some concluding remarks.

2 The model

This section develops a continuous piecewise linear OLG model featuring imperfect technology choice. Building on the framework of Umezuki and Yokoo (2019), we incorporate imperfect information to capture the idea that the threshold for technology choice is not perfectly known to firm owners and derive the piecewise linear form of the model.

2.1 Perfect technology choice

Time is discrete and extends from zero to infinity ($t = 0, 1, 2, ...$). The basic framework follows Diamond (1965). Each generation lives for two periods and supplies labor inelastically only in youth. The agent born at time t derives utility from consumption in the first and second periods of life, respectively. The utility function is assumed to take the log-linearized Cobb-Douglas form.

$$u(c_t^y, c_{t+1}^o) = (1-s)\ \log\ c_t^y + s\ \log\ c_{t+1}^o, \tag{1}$$

where $s \in (0,1)$ is the constant weight on consumption in old age, c_t^y denotes the consumption of the young, and c_{t+1}^o denotes that of the old.

The young agent supplies one unit of labor and receives the wage w_t for consumption and saving s_t. When the agent becomes old, consumption is based on his savings from the previous period. The budget constraints are

$$c_t^y + s_t = w_t, \quad \text{and} \quad c_{t+1}^o = r_{t+1} s_t, \tag{2}$$

where r_{t+1} represents the real interest rate. Maximizing utility given by (1) subject to constraints (2) yields

$$s_t = s w_t. \tag{3}$$

The final good Y is produced by the firm, which has two available production technologies. At the beginning of each period, the owner of the firm, who belongs to the old generation, chooses the technology that yields the highest return on capital. To illustrate this idea in a simple setting, we assume Cobb-Douglas production technologies

$$Y_t = F_i(K_t, L_t) = A_i K_t^{\alpha_i} L_t^{1-\alpha_i}, \qquad i \in \{1, 2\} \tag{4}$$

where K denotes capital, L denotes labor, $A_i > 0$ denotes total factor productivity, and $\alpha_i \in (0,1)$ is the capital share of the production technology i. Competition implies the following first-order conditions:

$$\begin{aligned} r_i(k_t) &= \alpha_i A_i k_t^{\alpha_i - 1}, \\ w_i(k_t) &= A_i(1-\alpha_i) k_t^{\alpha_i} \end{aligned} \tag{5}$$

where k is the capital-labor ratio. The firm's owners coordinate their choice of production technologies so that they earn the highest return on capital, or equivalently, capital income $kf'(k)$ given k. Then, this idea is given by

$$\max_{i\in\{1,2\}} r_i\left(k_t\right).$$

To avoid unnecessary subscripts, we rewrite the parameters as $(A_1, \alpha_1) = (A, \alpha)$ for technology 1 and $(A_2, \alpha_2) = (B, \beta)$ for technology 2. By solving $r_1(k) = r_2(k)$, we derive a unique threshold for technology choice,

$$c = \left[\frac{\alpha A}{\beta B}\right]^{\frac{1}{\beta-\alpha}}. \tag{6}$$

For simplicity and without affecting generality, we assume that $\alpha < \beta$. That is, technology 1 is chosen for $k \leq c$, and technology 2 is chosen for $k > c$, as illustrated in Fig. 1.

<< insert Fig. 1 around here >>

The market equilibrium can be represented by

$$k_{t+1} = s_t. \tag{7}$$

Using the optimization results above, we can obtain a one-dimensional dynamic model

$$k_{t+1} = T(k_t) = \begin{cases} s(1-\alpha)Ak_t^{\alpha} \equiv T_1(k_t), & \text{if} \quad 0 < k_t \leq c, \\ s(1-\beta)Bk_t^{\beta} \equiv T_2(k_t), & \text{if} \quad k_t > c. \end{cases} \tag{8}$$

The benchmark model represented by (8) exhibits periodic or non-periodic dynamics when suitable parameters are selected. However, the aperiodic behavior is restricted to a measure-zero set of parameters. This suggests that the model cannot explain the irregular or complex economic fluctuations often observed in reality. We address this limitation in the next subsection by considering observation errors due to agent heterogeneity.

2.2 Imperfect technology choice

Real world firm owners do not have perfect information about the economy and they differ in their observations of the true economic state due to agent heterogeneity. This implies that firm owners make the technology choice based on their individual observations. That is, the threshold for technology choice is not perfectly known to firm owners. To formalize this idea, we assume that the economic state is observed with some noise

$$\hat{k}_{i,t} = k_t + \sigma\varepsilon_{i,t}. \tag{9}$$

where k_t is perceived as $\hat{k}_{i,t}$ by the firm owner i at time t. And $\varepsilon_{i,t}$ is a zero-mean i.i.d disturbance term independently drawn from the cumulative distribution function, which will be specified later. The disturbance $\varepsilon_{i,t}$ represents the observational uncertainty by which the firm owner i is affected. We assume that the variance of the disturbance term is normalized to one, so that the degree of uncertainty is measured by σ.

In the case where $\sigma = 0$, the model relies on an unrealistic assumption that firm owners perceive the state variable in exactly the same way. We therefore restrict attention to the economically meaningful case $\sigma > 0$. In this case, the technology 1 is adopted when

$$\hat{k}_{i,t} \leq c \tag{10}$$

Then, by (9), inequality (10) can be rewritten as

$$\varepsilon_{i,t} \leq \frac{c - k_t}{\sigma} \equiv \rho(k_t) \tag{11}$$

Let G be the cumulative distribution function (CDF) for the stochastic variable ε. The probability of adopting technology 1 can be represented by

$$\text{Prob}\{\hat{k}_{i,t} \leq c\} = \text{Prob}\{\varepsilon_{i,t} \leq \rho(k_t)\} = G[\rho(k_t)].$$

Incorporating this probability into the original choice model (8), we obtain a general form of the model describing capital dynamics

$$k_{t+1} = G[\rho(k_t)]T_1(k_t) + (1 - G[\rho(k_t)])T_2(k_t). \tag{12}$$

where $G[\rho(k)]$ is necessarily bounded between 0 and 1. One can easily confirm that $G[\rho(k)] = 1$ yields $T_1(k)$ and $G[\rho(k)] = 0$ gives $T_2(k)$, which are two branches of the original model (8), respectively. To fully characterize the dynamics of (12), it is necessary to further specify the functional form of G. This is what we do in the next subsection.

2.3 Piecewise linear form of the model

We are now ready to clarify what distribution is required to ultimately obtain a piecewise linear economic model. [1] To facilitate the subsequent analysis, we begin by defining k_L and k_R as follows.

$$\rho(k_L) = 1, \qquad \rho(k_R) = -1.$$

[1] If we first assume that the disturbance term follows either a uniform or a logistic distribution, then the resultant model becomes a piecewise smooth model since two distinct technologies exist, that is, $\alpha \neq \beta$. Although this approach may be more intuitive, it is not ideal because it makes the subsequent analysis more difficult without contributing additional dynamics.

From (11), it follows that

$$k_L = c - \sigma, \qquad k_R = c + \sigma \tag{13}$$

Obviously, $\lim_{\sigma \to 0} k_L = \lim_{\sigma \to 0} k_R = c$ and $k_L < c < k_R$. Since capital per worker should be positive, we impose the condition that $0 < \sigma < c$. And k_L and k_R lie on either side of the original threshold value c. We can therefore take the interval

$$N = [T_2(k_R), T_1(k_L)] = [sB(1-\beta)k_R^{\beta},\ sA(1-\alpha)k_L^{\alpha}]$$

as a trapping interval for (12) if G is appropriately specified. To simplify the analysis of the dynamics, we transform the interval N into the unit interval $I = [0,1]$ via the variable change: $h : N \to I$:

$$x_t = h(k_t) = \frac{\log[k_t/T_2(k_R)]}{\log[T_1(k_L)/T_2(k_R)]}. \tag{14}$$

Then, the general form of the model given by (12) can be written as a piecewise map defined on the unit interval

$$x_{t+1} = \varphi(x_t) = \begin{cases} 1 + \alpha(x_t - \theta_L), & \text{if } 0 \le x_t \le \theta_L, \\ \varphi_M(x_t), & \text{if } \theta_L < x_t \le \theta_R, \\ \beta(x_t - \theta_R), & \text{if } \theta_R < x_t \le 1. \end{cases} \tag{15}$$

where

$$\theta_L = \theta_L(\sigma) = \frac{\log[k_L/T_2(k_R)]}{\log[T_1(k_L)/T_2(k_R)]}, \qquad \theta_R = \theta_R(\sigma) = \frac{\log[k_R/T_2(k_R)]}{\log[T_1(k_L)/T_2(k_R)]}, \tag{16}$$

where $\lim_{\sigma\to 0} \theta_L = \lim_{\sigma \to 0} \theta_R = \theta = \log[c/T_2(c)]/\log[T_1(c)/T_2(c)]$. For the purpose of constructing a model that is tractable and has rich dynamics, we set

$$\varphi_M(x_t) = \frac{\theta_R - x_t}{\theta_R - \theta_L}. \tag{17}$$

Based on this φ_M, we can derive the G_M. Let $y_t = \rho(k_t)$, then $0 \le G(y) \le 1$ for all $y \in \mathbb{R}$. For $-1 < y \le 1$, from the construction of φ_M, it follows that

$$G[\rho(k)]T_1(k) + (1 - G[\rho(k)])T_2(k) = h^{-1}(\varphi_M[h(k)]). \tag{18}$$

Solving the equation (18) for $G(y)$ yields

$$\begin{aligned} G(y) &= \frac{h^{-1}[\varphi_M(h[\rho^{-1}(y)])] - T_2[\rho^{-1}(y)]}{T_1[\rho^{-1}(y)] - T_2[\rho^{-1}(y)]} \\ &= \frac{T_2(k_R) \cdot \left[\frac{T_1(k_L)}{T_2(k_R)}\right]^{\frac{\log[k_R/(c-\sigma y)]}{\log[k_R/k_L]}} - T_2(c - \sigma y)}{T_1(c - \sigma y) - T_2(c - \sigma y)} \\ &\equiv G_M. \end{aligned} \tag{19}$$

It can be verified that $\lim_{y \to -1} G_M = 0$, $\lim_{y \to 1} G_M = 1$, and $G_M > 0$ for $y \in (-1, 1)$; see the Appendix for a detailed computation of G_M. Therefore, for $y \in \mathbb{R}$, $G(y)$ can be represented by

$$G(y) = \begin{cases} 0, & \text{if } \ y \leq -1, \\ G_M, & \text{if } \ -1 < y < 1, \\ 1, & \text{if } \ y \geq 1. \end{cases} \tag{20}$$

The graph of $G(y)$ is plotted in Fig. 2.

<< insert Fig. 2 around here >>

Accordingly, under the specified distribution, the main model analyzed in this paper is given by

$\varphi : I \to I$,

$$x_{t+1} = \varphi(x_t) = \begin{cases} 1 + \alpha(x_t - \theta_L) \equiv \varphi_L(x_t), & \text{if } 0 \leq x_t < \theta_L, \\ (\theta_R - x_t)/(\theta_R - \theta_L) \equiv \varphi_M(x_t) & \text{if } \theta_L \leq x_t < \theta_R, \\ \beta(x_t - \theta_R) \equiv \varphi_R(x_t), & \text{if } \theta_R \leq x_t \leq 1. \end{cases} \tag{21}$$

The map given by (21) is a deterministic model, as x_{t+1} is entirely determined by x_t. φ is N-shaped and has three regimes. The left and right regimes arise from the perfect technology choice, while the middle regime is induced by observation errors due to agent heterogeneity. Intuitively, the imperfect technology choice causes the emergence of the downward-sloping middle structure in the model.

3 Chaotic fluctuations

This section reveals that the model given by (21) can exhibit chaotic dynamics with the generic property, with the middle structure playing a crucial role in driving observable chaos. This feature overcomes the limitation of the model proposed by Umezuki and Yokoo (2019), in which "chaos" has a non-generic property; that is, in their model, aperiodic behavior is restricted to a measure-zero set of parameters and is thus not observable.

3.1 Markov property and observable chaos

Prior to the analysis, we recall some definitions of the Markov property. Let $I = [0, 1]$ and let $f : I \to I$ be a map of I onto itself. The interval I is divided into I_i by a finite partition $\mathcal{P}$. Define f_i as the restriction of f to I_i. If f_i is a homeomorphism from I_i onto some connected union of intervals of $\mathcal{P}$, then f is said to be Markov. The partition $\mathcal{P}$ is said to be a Markov partition with respect to f.

The model given by (21) features two kinks apart from the endpoints 0 and 1, which implies that the number of endpoints of the Markov partition needs to be strictly greater than 4. Therefore, we first show that $\varphi : I \to I$ has a Markov partition, whose endpoints form a period-5 cycle such that

$$0 = \varphi^5(0) < \theta_{5,L} = \varphi^2(0) < \theta_{5,R} = \varphi^4(0) < \gamma = \varphi(0) < \varphi^3(0) = 1, \tag{22}$$

This period-5 cycle requires that $\theta_{5,L}$ and $\theta_{5,R}$ satisfy

$$\varphi_R(\varphi_L(0)) = \theta_{5,L} \quad \text{and} \quad \varphi_R(1) = \theta_{5,R}.$$

Solving the equations above yields

$$\theta_{5,L} = \frac{\beta}{(1+\beta)(1+\alpha\beta)} \quad \text{and} \quad \theta_{5,R} = \frac{\beta}{1+\beta}. \tag{23}$$

Note that $\theta_{5,R} - \theta_{5,L} = \alpha\beta^2/[(1+\beta)(1+\alpha\beta)]$ and $\gamma = \varphi(0) = (1+\beta+\alpha\beta^2)/[(1+\beta)(1+\alpha\beta)]$. Direct calculations show that $\theta_{5,L}$ and $\theta_{5,R}$ are attained by choosing σ and s appropriately. In fact, using (13), (16) and (25) to solve

$$\theta_L(\sigma, s) = \theta_{5,L}, \quad \text{and} \quad \theta_R(\sigma, s) = \theta_{5,R}$$

for σ and s, we obtain special $\hat{\sigma}$ and $\hat{s}$. See the Appendix for the detailed derivations of $\hat{\sigma}$ and $\hat{s}$. For analytical convenience, the calculations are carried out under the condition $c = 1$; [2] then $\hat{s}$ can be expressed as

$$\hat{s} = s(\hat{\sigma}) = \frac{\beta}{\alpha A(1-\beta)} \left[\frac{(1-\hat{\sigma})^{1+\alpha\beta}}{(1+\hat{\sigma})^{1+\alpha\beta^2}} \right]^{\frac{1}{\alpha\beta}}.$$

Now, we demonstrate that the model given by (21) exhibits observable chaos on the period-5 Markov partition.

A piecewise smooth map $f : I \to I$ is said to be *(piecewise) eventually expanding* if $|(f^n)'(x)| \gg 1$ for integer $n \geq 1$ whenever the derivative exists. In addition, f is said to exhibit *observable chaos* if it has an ergodic invariant measure that is absolutely continuous with respect to Lebesgue measure. By the *Folklore Theorem*, it is known that an eventually expanding piecewise linear Markov map exhibits observable chaos. [3]

[2] Other values of c are possible, but they would make the analysis unnecessarily complex.

[3] Roughly speaking, the Folklore Theorem (Boyarsky and Gora, 1997, Theorem 6.1.1, p112) here says that if a map $f : I \to I$ satisfies "smoothness", "local invertibility", Markov property, "aperiodicity" and is piecewise expanding, then it exhibits observable chaos in the sense that it has an ergodic and hence unique absolutely continuous invariant measure.

Proposition 1. *(Observable chaos on a period-5 Markov partition) If $\theta_L = \theta_{5,L}$, $\theta_R = \theta_{5,R}$, $\sigma = \hat{\sigma} = \sigma(\theta_{5,L}, \theta_{5,R})$, and $s = \hat{s} = s(\hat{\sigma})$, then φ defined by* (21) *has a period-5 Markov partition of the unit interval and exhibits observable chaos.*

Proof. Let $I_1 = (0, \theta_{5,L})$, $I_2 = (\theta_{5,L}, \theta_{5,R})$, $I_3 = (\theta_{5,R}, \gamma)$, and $I_4 = (\gamma, 1)$. Then, one can easily confirm that

$$\varphi^3(I_1) = \varphi(I_2) = \varphi^4(I_3) = \varphi^2(I_4) = I. \tag{24}$$

Since every point $x \in \cup_{i=1}^4 I_i$ will visit I_1 at least once every fourth iteration, it will also visit I_2 at least once every fourth iteration. Therefore,

$$|(\varphi^4)'(x)| \geq |\varphi_L'| \cdot |\varphi_M'| \cdot |\varphi_R'|^2 = 1 + \beta + \alpha\beta + \alpha\beta^2 > 1. \tag{25}$$

whenever the derivatives exist. Thus, φ is eventually expanding, which implies that the map exhibits observable chaos.

□

Analogously, we can construct another Markov partition on a period-7 cycle such that

$$0 = \varphi^7(0) < \theta_{7,L} = \varphi^3(0) < \theta_{7,R} = \varphi^6(0) < \varphi^2(0) < \varphi^5(0) < \varphi(0) < \varphi^4(0) = 1.$$

Thus, solving

$$\varphi_R^2(\varphi_L(0)) = \theta_{7,L} \quad \text{and} \quad \varphi_R^2(1) = \theta_{7,R}$$

for $\theta_{7,L}$ and $\theta_{7,R}$, we obtain

$$\theta_{7,L} = \frac{\beta^2}{(1+\alpha\beta^2)(1+\beta+\beta^2)} \quad \text{and} \quad \theta_{7,R} = \frac{\beta^2}{1+\beta+\beta^2}.$$

Note also that $\theta_{7,R} - \theta_{7,L} = \alpha\beta^4/[(1+\alpha\beta^2)(1+\beta+\beta^2)]$.

We can readily extend proposition 1 to a more general case of a period-$(2n+3)$ Markov partition for $n \geq 1$.

Proposition 2. *(Observable chaos on a period-(2n+3) Markov partition) If $\theta_L = \theta_{2n+3,L}$, $\theta_R = \theta_{2n+3,R}$, $\sigma = \hat{\sigma} = \sigma(\theta_{2n+3,L}, \theta_{2n+3,R})$, and $s = \hat{s} = s(\hat{\sigma})$, then φ defined by* (21) *has a period-(2n+3) Markov partition of the unit interval and exhibits observable chaos.*

Proof. By solving

$$\varphi_R^n(\varphi_L(0)) = \theta_{2n+3,L} \quad \text{and} \quad \varphi_R^n(1) = \theta_{2n+3,R}.$$

We can derive

$$\theta_{2n+3,L} = \frac{\beta^n(1-\beta)}{(1-\beta^{n+1})(1+\alpha\beta^n)} \quad \text{and} \quad \theta_{2n+3,R} = \frac{\beta^n(1-\beta)}{1-\beta^{n+1}}. \tag{26}$$

Note that $\theta_{2n+3,R} - \theta_{2n+3,L} = \alpha\beta^{2n}(1-\beta)/[(1-\beta^{n+1})(1+\alpha\beta^n)]$. By an analogous argument, we obtain special $\hat{\sigma}$ and $\hat{s}$ by solving $\theta_L(\sigma, s) = \theta_{2n+3,L}$ and $\theta_R(\sigma, s) = \theta_{2n+3,R}$ for σ and s. See the appendix for the detailed derivations.

Since every point $x \in \cup_{i=1}^{2n+2} I_i$ will visit I_1 at least once every $(2n+2)$th iteration, it will also visit I_2 at least once every $(2n+2)$th iteration, we have

$$|(\varphi^{2n+2})'(x)| \geq |\varphi'_L| \cdot |\varphi'_R|^{2n} \cdot |\varphi'_M| = (1+\alpha\beta^n)(1+\beta+\cdots+\beta^n) > 1. \tag{27}$$

whenever the derivatives exist. Therefore, φ is eventually expanding and exhibits observable chaos on a period-$(2n+3)$ Markov partition.

□

The period-5 Markov property for $n = 1$ is illustrated in Figs. 3 and 4. Fig. 3 shows the dynamics starting from the periodic point, whereas Fig. 4 shows the dynamics starting from a non-periodic point. The difference in patterns between the two figures also indicates an interesting feature of chaos, namely sensitive dependence on initial conditions.

<< insert Fig. 3 around here >>

<< insert Fig. 4 around here >>

These results suggest that observable chaos is not confined to a special case but arises in general cases. Thus, our deterministic framework has captured long-run chaotic fluctuations, which allows for the reproduction of recurrent but non-periodic behavior often observed in real business cycle fluctuations.

4 Ergodic Markov chains

This section shows that chaotic dynamics established previously can be described by stationary distributions, and its transition patterns between different economic states follow ergodic Markov chains.

4.1 Stationary distributions of chaotic dynamics

When the piecewise linear model is Markov, the invariant measures, also known as stationary distributions, can be derived explicitly. This allows us to obtain the long-run averages for the chaotic dynamics. See, for example, Huang and Day (1993) or Matsumoto (2005) for the derivation of invariant densities in other economic contexts.

We derive invariant densities using a matrix version of the Frobenius-Perron Operator, $M_\varphi = (m_{ij})_{1\le i,j\le n}$, the entries of which are given by

$$m_{ij} = \frac{q_{ij}}{|\varphi_i'|}, \qquad 1 \le i, j \le n,$$

where $Q_\varphi = (q_{ij})_{1\le i,j\le n}$ is the incidence matrix induced by the piecewise monotonic transformation φ and the partition $\mathcal{P}$ of I. In particular,

$$q_{ij} = \begin{cases} 1, & \text{if} \quad I_j \subset \varphi(I_i), \\ 0, & \text{otherwise.} \end{cases}$$

For further details, see, for example, Boyarsky and Góra(1997) or Boyarsky and Scarowsky (1979). If φ is a piecewise linear Markov transformation and $\inf|(\varphi^k)'(x)| > 1$ for some $k \ge 1$ wherever the derivative exists, then φ admits a unique invariant density function, which is the nontrivial solution of $\pi M_\varphi = \pi$, that is, the left eigenvector for eigenvalue 1.

Now, let us first examine the simplest period-5 Markov partition of our model.

Proposition 3. *(Invariant density on a period-5 Markov partition) If the assumptions stated in proposition 1 are satisfied, then φ admits a unique invariant density function $\pi = \pi_5^*(x)$ that is piecewise constant on $\{I_i\}_{i=1}^4$, and the corresponding probability density is $p(x) = \pi_5^*(x)/\sum_{i=1}^4 \pi_i|I_i|$.*

Proof. On the period-5 Markov partition, we observe that

$$I_4 \subset \varphi(I_1), \quad \cup_{i=1}^4 I_i \subset \varphi(I_2), \quad I_1 \subset \varphi(I_3), \quad I_2 \subset \varphi(I_4).$$

Since

$$|\varphi_1'| = \alpha, \quad |\varphi_2'| = \frac{1}{\theta_{5,R} - \theta_{5,L}} = \frac{(1+\beta)(1+\alpha\beta)}{\alpha\beta^2}, \quad |\varphi_3'| = |\varphi_4'| = \beta,$$

we obtain

$$M_{\varphi_5} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\alpha} \\ \triangle & \triangle & \triangle & \triangle \\ \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 \end{pmatrix},$$

where $\triangle = \alpha\beta^2/[(1+\beta)(1+\alpha\beta)]$. By Theorem 9.4.2 of Boyarsky and Gora(1997), φ admits an invariant density, which is the nontrivial solution of $\pi M_{\varphi_5} = \pi$, thus

$$\pi = \pi_5^*(x) = \begin{cases} \alpha\beta + \alpha\beta^2 \equiv \pi_1, & x \in I_1 \\ \alpha\beta^2 + \alpha\beta + 1 + \beta \equiv \pi_2, & x \in I_2 \\ \alpha\beta^2 \equiv \pi_3, & x \in I_3 \\ \alpha\beta^2 + \beta + \beta^2 \equiv \pi_4, & x \in I_4 \end{cases}. \tag{28}$$

$\square$

Fig. 5 illustrates the probability density derived above, with the area for intervals I_i corresponding to the frequency that the chaotic system visits each interval. This result implies how often a chaotically fluctuating economy visits a particular economic state in the long run.

<< insert Fig. 5 around here >>

The above argument extends naturally to higher periods.

Proposition 4. *(Invariant density on a period-(2n+3) Markov partition) If the assumptions stated in proposition 2 are satisfied, then φ admits a unique invariant density function $\pi = \pi^*_{2n+3}(x)$ that is piecewise constant on $\{I_i\}_{i=1}^{2n+2}$ and its corresponding probability density is $p(x) = \pi^*_{2n+3}(x)/\sum_{i=1}^{2n+2}\pi_i|I_i|$.*

Proof. By the same argument, we obtain

$$M_{\varphi_{2n+3}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{1}{\alpha} \\ \triangle & \triangle & \cdots & \triangle & \triangle & \triangle \\ \frac{1}{\beta} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\beta} & 0 & 0 \end{pmatrix}_{(2n+2)\times(2n+2)}$$

where $\triangle = \theta_{2n+3,R} - \theta_{2n+3,L} = \alpha\beta^{2n}(1-\beta)/[(1-\beta^{n+1})(1+\alpha\beta^n)]$. Then, the invariant density $\pi^*_{2n+3}(x)$ is derived as follows.

$$\pi = \pi^*_{2n+3}(x) = (\pi_1, \pi_2, \cdots, \pi_{2n+1}, \pi_{2n+2}).$$

In this case, $\pi_2 = (1+\alpha\beta^n)(1+\beta+\cdots+\beta^n)$ represents the maximum, while $\pi_{2n+1} = \alpha\beta^{2n}$ denotes the minimum. The remaining terms follow from these two.

□

The invariant density indicates the long-run average distribution of distinct economic states of the chaotic economic system. The result is useful because economic fluctuations that seem highly unpredictable display the predictable feature that underlying distribution patterns can be explicitly expressed.

4.2 Transition patterns through a Markov chain

To gain further insights, we revisit the matrix M_{φ_5} obtained above. By normalizing the 4×4 matrix M_{φ_5}, the sum of all entries in each row equals 1; that is, $\sum_{j=1}^{4} p_{ij} = 1$.

Then the row-normalized matrix $P_{\varphi_5} = (p_{ij})_{1<i,j<4}$ can be expressed as

$$P_{\varphi_5} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

where $p_{ij} = |I_i|/|I| = |I_i|$, with $a_1 = \beta/(1+\beta+\alpha\beta+\alpha\beta^2)$, $a_2 = \alpha\beta^2/(1+\beta+\alpha\beta+\alpha\beta^2)$, $a_3 = 1/(1+\beta+\alpha\beta+\alpha\beta^2)$, $a_4 = \alpha\beta/(1+\beta+\alpha\beta+\alpha\beta^2)$. It follows that $a_1+a_2+a_3+a_4 = 1$.

If the state space consists of the interval I_1, I_2, I_3 and I_4, with each row representing the current state and each column representing the next state, then the entries of the matrix represent the transition probability from the row to the column, namely, from the current state to the next one.

After two periods, the transition matrix can be given by

$$P^2_{\varphi_5} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}.$$

where $b_1 = a_1a_2 + a_3$, $b_2 = a_2^2 + a_4$, $b_3 = a_2a_3$, and $b_4 = a_1 + a_2a_4$. After n periods, the transition pattern can be represented by the n-th power of the matrix $P^n_{\varphi_5}$. Further calculations show that $P^n_{\varphi_5}$ has strictly positive entries for $n = 4$. It follows that P_{φ_5} is a transition matrix of a Markov chain. As shown in Fig. 6, the transition matrix of the Markov chain is depicted in a state transition diagram. The figure shows that it is possible to get from any state to any other state, independent of the initial state; hence, the Markov chain is an ergodic Markov chain.

<< insert Fig. 6 around here >>

Proposition 5. *(Stationary distribution of a Markov chain) If the assumptions stated in proposition 1 are satisfied, then the long-term distribution of chaotic dynamics for the map φ follows a Markov chain with a transition matrix $P = P_{\varphi_5}$.*

Proof. Let y be a left eigenvector of the transition matrix P_{φ_5} with eigenvalue 1. Solving $yP_{\varphi_5} = y$ yields

$$y = (y_1, y_2, y_3, y_4).$$

The detailed derivation and explicit expressions are given in the Appendix. Let $S_i = \pi_i|I_i|$ denote the areas of the corresponding intervals. A direct computation shows that

$$y_i = \lambda S_i, \quad i = 1, 2, 3, 4$$

where the explicit expression of $\lambda > 0$ is provided in the Appendix. Hence, the stable distribution vector of the transition matrix is proportional to the areas of the finite intervals. $\square$

By an analogous argument, the case of the period-7 Markov partition follows. The transition matrix is represented by P_{φ_7}, and solving $yP_{\varphi_7} = y$ yields

$$y = (y_1, y_2, y_3, y_4, y_5, y_6).$$

Then, by calculating $S_i = \pi_i |I_i|$, we find that $y_i = \lambda S_i$ still holds. The details are given in the Appendix.

This argument readily extends to the general setting. The row-normalized matrix $P_{\varphi_{2n+3}}$ can be expressed as

$$P_{\varphi_{2n+3}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}_{(2n+2)\times(2n+2)}$$

where $p_{ij} = |I_i|/|I| = |I_i|$ and $\sum_{j=1}^{2n+2} p_{ij} = 1$. Thus, each entry denotes the transition probability from interval i to interval j. Similarly, after n periods, the transition pattern can be characterized by the power of the matrix $P^n_{\varphi_{2n+3}}$, which can be used to predict future economic states. We conjecture that all entries of $P^{2n+2}_{\varphi_{2n+3}}$ are positive.

Proposition 6. *(The general case of Proposition 5) If the assumptions stated in proposition 2 are satisfied, then the long-term distribution of chaotic dynamics for the map φ follows a Markov chain with a transition matrix $P = P_{\varphi_{2n+3}}$.*

Proof. See Appendix. $\square$

Therefore, the analysis reveals a seemingly paradoxical feature that deterministic chaos, which is sensitive to the initial value, can be described by a Markov chain, which is independent of it. Specifically, the original model is deterministic; thus, once the initial values are determined, all subsequent states are decided. By contrast, the Markov chain shows the memoryless property that the next state depends only on the current state. This implies that the Markov chain provides a counterintuitive insight into the long-run and ergodic behavior of the chaotic economic system.

Moreover, an increase in the number of subintervals allows for a more precise distinction among different economic states, indicating that transition patterns between economic states might become closer to the actual situation. From such transition patterns of an ergodic Markov chain, one may alternatively infer how many periods it will take for the chaotically fluctuating economy to enter a recession phase if it is currently in a boom phase. This interpretation, while tentative, suggests that the result has a certain degree of predictive power.

5 Robustness

So far, we have seen that invariant measures are piecewise constant under the specific distribution of observation errors. We now relax the distributional restriction and show that the invariant measure maintains a certain robustness.

We first define the Skorokhod metric. Let $I = [0,1]$ with normalized Lebesgue measure λ. Let $\mathcal{T}(I)$ denote the class of all piecewise expanding transformations. The Skorokhod metric on $\mathcal{T}(I)$ is defined as follows: for $\tau_1, \tau_2 \in \mathcal{T}$,

$$\begin{aligned} d_S(\tau_1, \tau_2) = \inf \big\{ \varepsilon > 0 : \exists A \subseteq I \text{ and } \exists\ \xi : I \to I \text{ such that} \\ \lambda(A) > 1 - \varepsilon,\ \xi \text{ is a diffeomorphism,} \\ \tau_{1|A} = \tau_2 \circ \tau|_A \text{ and } \forall x \in A,\ |\xi(x) - x| < \varepsilon,\ |\frac{1}{\xi'(x)} - 1| < \varepsilon \big\}. \end{aligned}$$

Theorem (Boyarsky & Góra , 1997, Theorem 11.2.2). Let $\tau, \tau_n \in \mathcal{T}(I)$. Suppose that there exist $\delta > 0$ and $0 < r < 1$ such that for any $\eta = \tau$ or $\tau_n, n = 1, 2, ...,$

(a) $\min_{1 \leqslant i \leqslant q} \lambda \left(I_i^{(\eta)} \right) \geqslant \delta$,

(b) $2 \left(\inf_{x \in I} |\eta'(x)| \right)^{-1} + \max_{1 \leqslant i \leqslant q} V_{I_i^{(\eta)}} \left(|\eta'(x)|^{-1} \right) \leqslant r$,

where $V_K f$ denotes the total variation of function f on interval K. If τ is ergodic and $d_S(\tau, \tau_n) \to 0$ as $n \to \infty$, then τ_n is ergodic for n large enough and $\pi_n \to \pi$ in $\mathfrak{L}^1$ as $n \to \infty$,π_n and π are invariant densities of τ_n and τ, respectively.

Let ε be a small positive number, and φ_ε be a perturbation of φ. Let $\{\varphi_n\}$ be a sequence corresponding to $\{\varepsilon_n : \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > 0\}$. To show the stability of invariant densities, it suffices to prove the convergence property of the sequence $\{\varphi_n\}$ in the Skorokhod metric. It can be confirmed that φ or φ_n satisfies the conditions (a) and (b) in Theorem 11.2.2 since they are piecewise linear. We restrict our attention to the interval $[\theta_L, \theta_R]$.[4]

[4]From the perspective of the model's structure, the left and right branches governed by two technologies remain unchanged; only the middle branch influenced by heterogeneous observations is perturbed deterministically. Although analyzing the full interval I is also feasible, it would introduce unnecessary complexity. For analytical convenience, we focus on the interval $[\theta_L, \theta_R]$.

Writing φ_M in this form and defining φ_ε accordingly, we obtain

$$\varphi_M(x) = \frac{\theta_R - x}{\theta_R - \theta_L} = \begin{cases} \dfrac{(1-\theta_L)\bar{x} - (1-\bar{x})x}{\bar{x} - \theta_L}, & \text{if } \theta_L \le x < \bar{x}, \\ \dfrac{-\bar{x}(x-\theta_R)}{\theta_R - \bar{x}}, & \text{if } \bar{x} \le x \le \theta_R. \end{cases}$$

$$\varphi_\varepsilon(x) = \begin{cases} \dfrac{(1-\theta_L)\bar{x} - (1+a\varepsilon-\bar{x})x}{\bar{x} - \theta_L(1+a\varepsilon)}, & \text{if } \theta_L \le x < \frac{\bar{x}}{1+a\varepsilon}, \\ \dfrac{-\bar{x}(x-\theta_R)}{(1+a\varepsilon)\theta_R - \bar{x}}, & \text{if } \frac{\bar{x}}{1+a\varepsilon} \le x \le \theta_R. \end{cases} \tag{29}$$

where a is a technical parameter. The construction is illustrated in Fig. 7.

<< insert Fig. 7 around here >>

Proposition 7. *(Robustness of invariant measures) Let the assumptions stated in proposition 1 be satisfied, so that φ satisfies the Markov property. If the perturbation φ_ε satisfies* (29) *, then the unique invariant densities of $\{\varphi_n\}$ converge in $\mathfrak{L}^1$ to the unique invariant density of φ.*

Proof. To estimate the Skorokhod metric $d_S(\varphi_\varepsilon, \varphi_M)$, let $\bar{x} > 0$ be such that $\varphi_M(\bar{x}) = \bar{x}$ and $A = A_1 \cup A_2$, where $A_1 = [\theta_L, \bar{x}/(1+a\varepsilon)]$ and $A_2 = [\bar{x}, \theta_R]$. We first construct a diffeomorphism $\xi(x)$ such that $\varphi_\varepsilon = \varphi_M \circ \xi$ on A.

On A_1, we obtain

$$\xi_{|A_1}(x) = \xi_1(x) = \frac{(1+a\varepsilon-\bar{x})(\bar{x}-\theta_L)\ x - a\varepsilon\theta_L(1-\theta_L)\bar{x}}{[\bar{x} - (1+a\varepsilon)\theta_L](1-\bar{x})}.$$

In the same way, on A_2, we have

$$\xi_{|A_2}(x) = \xi_2(x) = \frac{(\theta_R - \bar{x})\ x + a\varepsilon\theta_R^2}{\theta_R(1+a\varepsilon) - \bar{x}}.$$

To construct ξ on $(\bar{x}/(1+a\varepsilon), \bar{x})$, we define ξ' to be piecewise linear and then define ξ as the integral of ξ'. We then have $\xi(\bar{x}/(1+a\varepsilon))$ and $\xi(\bar{x})$. Thus, the average value of ξ' on $(\bar{x}/(1+a\varepsilon), \bar{x})$ is given by

$$\xi'_{\text{average}} = \frac{|\xi(\bar{x}) - \xi(\bar{x}/(1+a\varepsilon))|}{a\varepsilon\bar{x}/(1+a\varepsilon)}.$$

Also

$$\xi'_L = \xi'\left(\frac{\bar{x}}{1+a\varepsilon}\right) = \frac{(1+a\varepsilon-\bar{x})(\bar{x}-\theta_L)}{[\bar{x}-(1+a\varepsilon)\theta_L](1-\bar{x})} \qquad \text{and}$$

$$\xi'_R = \xi'(\bar{x}) = \frac{\theta_R - \bar{x}}{\theta_R(1+a\varepsilon) - \bar{x}}$$

Let us construct the graph of ξ' on $[\bar{x}/(1+a\varepsilon), \bar{x}]$ as follows: Draw it as a continuous piecewise linear curve, whose endpoints are given by ξ'_L and ξ'_R, such that the two regions enclosed by this curve and the line given by ξ'_{average}, one region above this line and the other below it have equal areas. See Boyarsky and Góra (1997, p.233) for details.

Next, we estimate $d_S(\varphi_1, \varphi_2)$. Let a be a sufficiently small positive number.[5]

On $A_1 = [\theta_L, \bar{x}/(1+a\varepsilon)]$, we have

$$|\xi_1(x) - x| = \left| \frac{a\varepsilon(1-\theta_L)\bar{x}(x-\theta_L)}{[\bar{x} - (1+a\varepsilon)\theta_L](1-\bar{x})} \right| \leq \frac{a(1-\theta_L)\bar{x}\varepsilon}{1-\bar{x}} \leq \varepsilon.$$

$$\left| \frac{1}{\xi'_1(x)} - 1 \right| = \left| \frac{a\varepsilon\bar{x}(\theta_L - 1)}{(1+a\varepsilon-\bar{x})(\bar{x}-\theta_L)} \right| \leq \frac{a\bar{x}(1-\theta_L)\varepsilon}{(1-\bar{x})(\bar{x}-\theta_L)} \leq \varepsilon.$$

On $A_2 = [\bar{x}, \theta_R]$, we have

$$|\xi_2(x) - x| = \left| \frac{a\varepsilon\theta_R(\theta_R - x)}{\theta_R(1+a\varepsilon) - \bar{x}} \right| < a\theta_R\varepsilon \leq \varepsilon.$$

$$\left| \frac{1}{\xi'_2(x)} - 1 \right| = \frac{a\theta_R\varepsilon}{\theta_R - \bar{x}} \leq \varepsilon.$$

Therefore, $d_S(\varphi_\varepsilon, \varphi_M) \leq \varepsilon$ for ε small enough on $A = A_1 \cup A_2$. This result indicates that $d_S(\varphi_n, \varphi) \to 0$ as $n \to +\infty$. Since φ is ergodic, it follows that φ_n is ergodic for n large enough. Let $\{\pi_n\}$ be a sequence of invariant densities corresponding to $\{\varepsilon_n : \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > 0\}$. By Theorem 11.2.2 of Boyarsky and Góra (1997), unique invariant densities of $\{\varphi_n\}$ converge in $\mathfrak{L}^1$ to the unique invariant density of φ. $\square$

To intuitively understand the results above, it is useful to draw a bifurcation diagram with respect to ε, as shown in Fig. 8. At $\varepsilon = 0$, the model exhibits no perturbation, corresponding to the case in which the observation error follows the distribution function G given by (20).

As ε increases, $\bar{x}/(1+a\varepsilon)$ gradually moves away from $\bar{x}$, and the initially straight line φ_M becomes bent, forming two intersecting branches φ_ε. From Fig. 8, observable chaos persists for all $\varepsilon > 0$ near zero.

<< insert Fig. 8 around here >>

This indicates not merely that observable chaos persists under perturbations, but also that the stationary distributions of chaotic fluctuation patterns remain stable against perturbations related to observation errors.

[5] To be more specific, it suffices to set

$$a = \min\left\{ \frac{1-\bar{x}}{(1-\theta_L)\bar{x}}, \frac{(1-\bar{x})(\bar{x}-\theta_L)}{\bar{x}(1-\theta_L)}, \frac{1}{\theta_R}, \frac{\theta_R - \bar{x}}{\theta_R} \right\}.$$

6 Concluding remarks

This paper introduces imperfect observation into the OLG model with endogenous technology choice by Umezuki and Yokoo (2019), demonstrating how imperfect technology choice can induce chaotic macroeconomic fluctuations. The Umezuki and Yokoo (2019) model can exhibit periodic fluctuations of arbitrary periods, but its nonlinearity is not deep enough to generate chaos. In our model, a small "trembling hand" effect due to observation errors generates deep nonlinearity that randomizes the economy near the threshold.

However, the focus of this paper is not on introducing this mechanism, but rather on analyzing and interpreting the observable chaos captured by the OLG model. Specifically, it has been found that this complex phenomenon can be expressed through the transition patterns and stationary state distributions of Markov chains. Although it may seem counterintuitive, this approach to describing chaotic dynamics should be emphasized because it enables the application of probabilistic methods to the study of deterministic frameworks, thereby revealing the statistical properties of endogenous economic fluctuations.

Finally, the shape of the distribution function for observational errors appears to be subject to very stringent constraints in order to satisfy piecewise linearity. That is indeed the case. However, we have observed that the invariant densities for chaotic behavior are robust to certain perturbations of the distribution function for these observation errors. We expect that the approach developed in this study will be applied to the analysis of various economic models exhibiting complex endogenous fluctuations.

Appendix A. Computation of G_M

Substituting (14) into (17) yields

$$\varphi_M[h(k_t)] = \frac{\log[k_R/k_t]}{\log[k_R/k_L]}.$$

From the inverse of (14), we obtain

$$h^{-1}(\varphi_M[h(k_t)]) = T_2(k_R)\left[\frac{T_1(k_L)}{T_2(k_R)}\right]^{\frac{\log[k_R/k_t]}{\log[k_R/k_L]}}.$$

Then, rearranging (18) and substituting $h^{-1}(\varphi_M[h(k_t)])$ into it, we have

$$G[\rho(k)] = \frac{h^{-1}(\varphi_M[h(k)]) - T_2(k)}{T_1(k) - T_2(k)} = \frac{T_2(k_R) \cdot \left[\frac{T_1(k_L)}{T_2(k_R)}\right]^{\frac{\log[k_R/k]}{\log[k_R/k_L]}} - T_2(k)}{T_1(k) - T_2(k)} \equiv G_M.$$

Using $k = \rho^{-1}(y)$, $G[\rho(k)]$ can be expressed as a function of y, as shown in (19).

To verify $G_M > 0$ for $-1 < y < 1$, we examine the numerator and denominator separately for $k_L < k < k_R$. The denominator is obviously positive with this range of k. On the other hand, given that $T_2(k_R) > T_2(k)$ holds for $k_L < k < k_R$, to show the numerator is positive, it suffices to prove

$$\left[\frac{T_1(k_L)}{T_2(k_R)}\right]^{\frac{\log[k_R/k]}{\log[k_R/k_L]}} > 1.$$

Since $T_1(k_L)/T_2(k_R) > 1$ and $0 < \log(k_R/k)/\log(k_R/k_L) < 1$, the above inequality holds.

Appendix B. Computation of $\hat{\sigma}$ and $\hat{s}$

From (16) and (23), solving $\theta_L(\sigma) = \theta_{5,L}$ and $\theta_R(\sigma) = \theta_{5,R}$ reduces to solving

$$\frac{k_L}{T_2(k_R)} = \left[\frac{T_1(k_L)}{T_2(k_R)}\right]^{\frac{\beta}{(1+\alpha\beta)(1+\beta)}} \quad \text{and} \quad \frac{k_R}{T_2(k_R)} = \left[\frac{T_1(k_L)}{T_2(k_R)}\right]^{\frac{\beta}{1+\beta}}. \tag{30}$$

Using (13) and eliminating the parameter s from the above equations, we obtain the following equation in terms of σ. Given α, β, A, B, $\hat{\sigma}$ can then be attained by solving

$$\frac{c+\sigma}{c-\sigma} = \left[\frac{A(1-\alpha)(c-\sigma)^{\alpha}}{B(1-\beta)(c+\sigma)^{\beta}}\right]^{\frac{\alpha\beta^2}{(1+\alpha\beta)(1+\beta)}}$$

For analytical convenience, we set $c = 1$ (so that $\alpha A = \beta B$). Simplifying the above expression yields

$$(1+\sigma)^{1+\beta+\alpha\beta+\alpha\beta^2+\alpha\beta^3} = \left[\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right]^{\alpha\beta^2} (1-\sigma)^{1+\beta+\alpha\beta+\alpha\beta^2+\alpha^2\beta^2}.$$

Since $\alpha < \beta$, it follows that $[\beta(1-\alpha)]/[\alpha(1-\beta)] > 1$. Therefore, a solution for $\hat{\sigma}$ necessarily exists. Based on this $\hat{\sigma}$, we rearrange (30) and solve for $\hat{s}$, obtaining

$$\hat{s} = s(\hat{\sigma}) = \frac{\beta}{\alpha A(1-\beta)} \left[\frac{(1-\hat{\sigma})^{1+\alpha\beta}}{(1+\hat{\sigma})^{1+\alpha\beta^2}}\right]^{\frac{1}{\alpha\beta}}.$$

For the general case, we solve $\theta_L(\sigma, s) = \theta_{2n+3,L}$ and $\theta_R(\sigma, s) = \theta_{2n+3,R}$ for $\hat{\sigma}$ and $\hat{s}$ to find that $\hat{\sigma}$ can be obtained from

$$(1+\sigma)^{\mu} = \left[\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right]^{\alpha\beta^{2n}(1-\beta)} (1-\sigma)^{\nu}.$$

where $\mu = (1-\beta^{n+1})(1+\alpha\beta^n)+\alpha\beta^{2n+1}(1-\beta)$ and $\nu = (1-\beta^{n+1})(1+\alpha\beta^n)+\alpha^2\beta^{2n}(1-\beta)$. Based on this $\hat{\sigma}$, we have

$$\hat{s} = s(\hat{\sigma}) = \frac{\beta}{\alpha A(1-\beta)} \left[\frac{(1-\hat{\sigma})^{1+\alpha\beta^n}}{(1+\hat{\sigma})^{1+\alpha\beta^{n+1}}}\right]^{\frac{1}{\alpha\beta^n}}.$$

Appendix C. Proof of Proposition 5

Solving $yP_{\varphi_5} = y$ yields the following expressions:

$$y_1 = \frac{1+\beta}{4+3\beta+2\alpha\beta+\alpha\beta^2}, \qquad y_2 = \frac{1+\alpha\beta+\alpha\beta^2+\beta}{4+3\beta+2\alpha\beta+\alpha\beta^2},$$
$$y_3 = \frac{1}{4+3\beta+2\alpha\beta+\alpha\beta^2}, \qquad y_4 = \frac{1+\beta+\alpha\beta}{4+3\beta+2\alpha\beta+\alpha\beta^2}.$$

A straightforward calculation yields

$$\frac{y_i}{S_i} = \lambda = \frac{1+\alpha\beta+\alpha\beta^2+\beta}{\alpha\beta^2(4+3\beta+2\alpha\beta+\alpha\beta^2)}, \quad i = 1,2,3,4.$$

Appendix D. The case $n = 2$ and Proposition 6

The argument is analogous to that of Proposition 5. We briefly indicate the explicit expressions required for $n = 2$.

$$P_{\varphi_7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

where $p_{ij} = |I_i|/|I|$. Solving $yP_{\varphi_7} = y$ for y_i and calculating S_i yields

$$\frac{y_i}{S_i} = \lambda = \frac{1+\beta+\beta^2+\alpha\beta^2+\alpha\beta^3+\alpha\beta^4}{\alpha\beta^4(6+5\beta+4\beta^2+3\alpha\beta^2+2\alpha\beta^3+\alpha\beta^4)}, \quad i = 1,2,3,4,5,6.$$

We then outline the general case. Solving $yP_{\varphi_{2n+3}} = y$ yields

$$y = (y_1, y_2, \cdots, y_{2n+1}, y_{2n+2}).$$

Let $S_i = \pi_i|I_i|$ denote the areas of the corresponding intervals. A similar argument suggests that, in the general case, there exists a constant $\lambda > 0$ such that

$$y_i = \lambda S_i, \quad i = 1,2,\cdots,2n+2.$$

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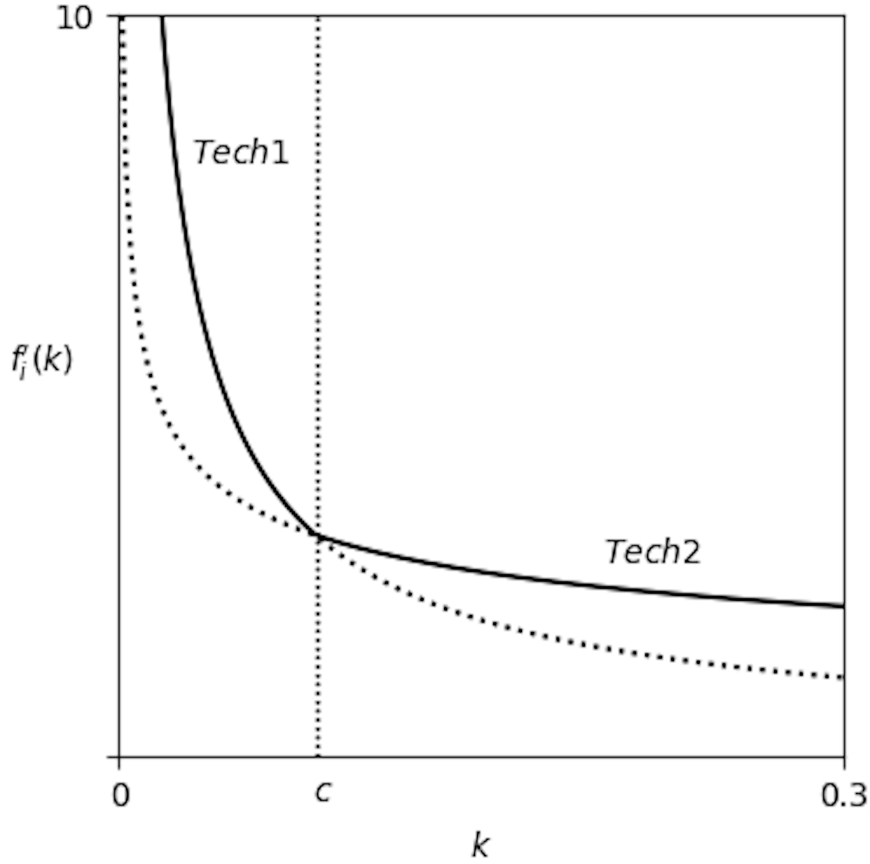

Figure 1: Technology choice. Solid lines indicate the chosen technologies. $\alpha = 0.2$, $\beta = 0.7$, $A = 2$, $B = 2$.

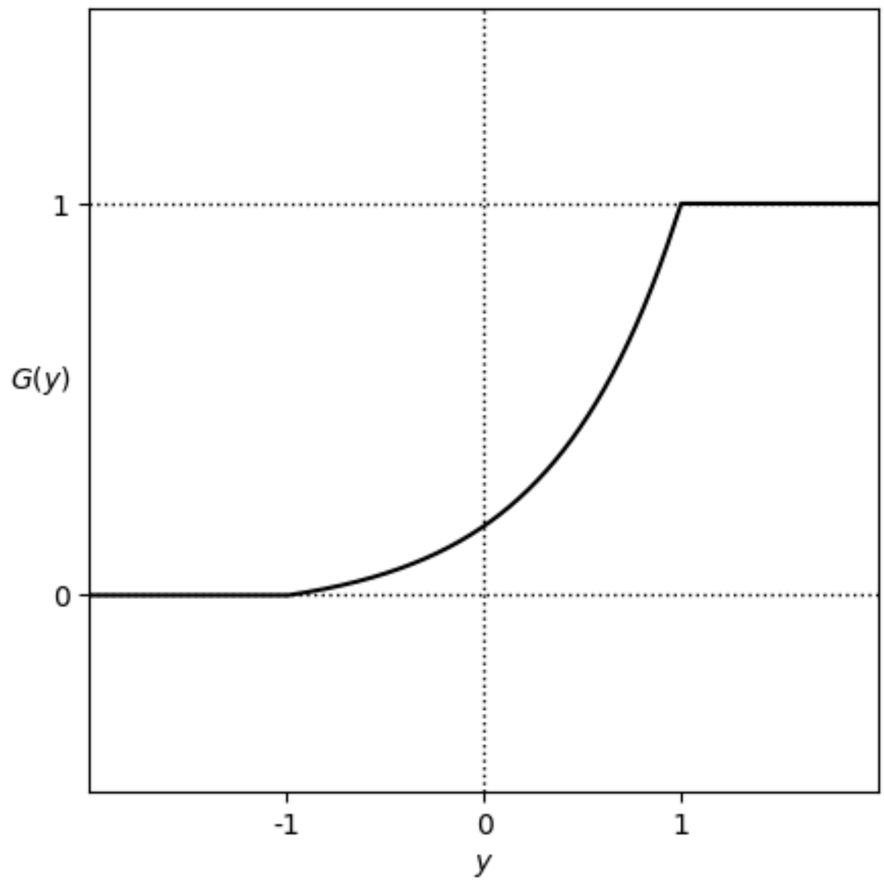

Figure 2: Graph of cumulative distribution function $G(y)$. $\alpha = 0.1$, $\beta = 0.7$, $A = 2$, $B = (\alpha A)/\beta$, $\sigma = 0.01$.

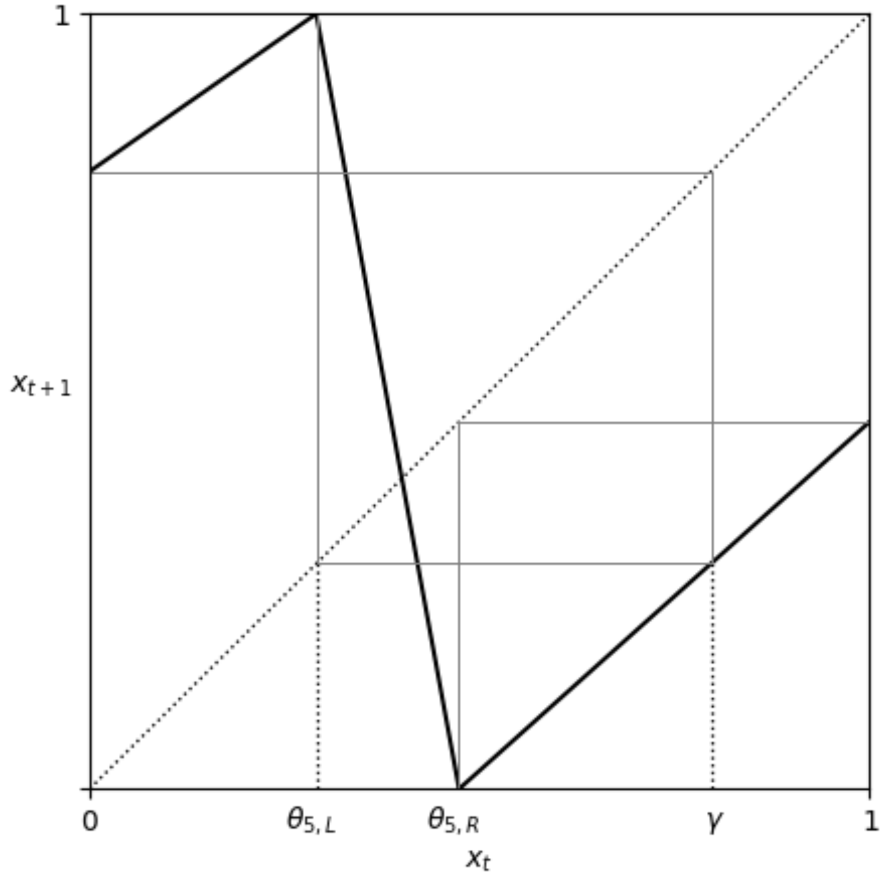

Figure 3: Period-5 Markov property and dynamics starting from the periodic point. $\alpha = 0.7$, $\beta = 0.9$, $A = 20$, $B = (\alpha A)/\beta$, $\hat{\sigma} \approx 0.1075$, $\hat{s} \approx 0.372$.

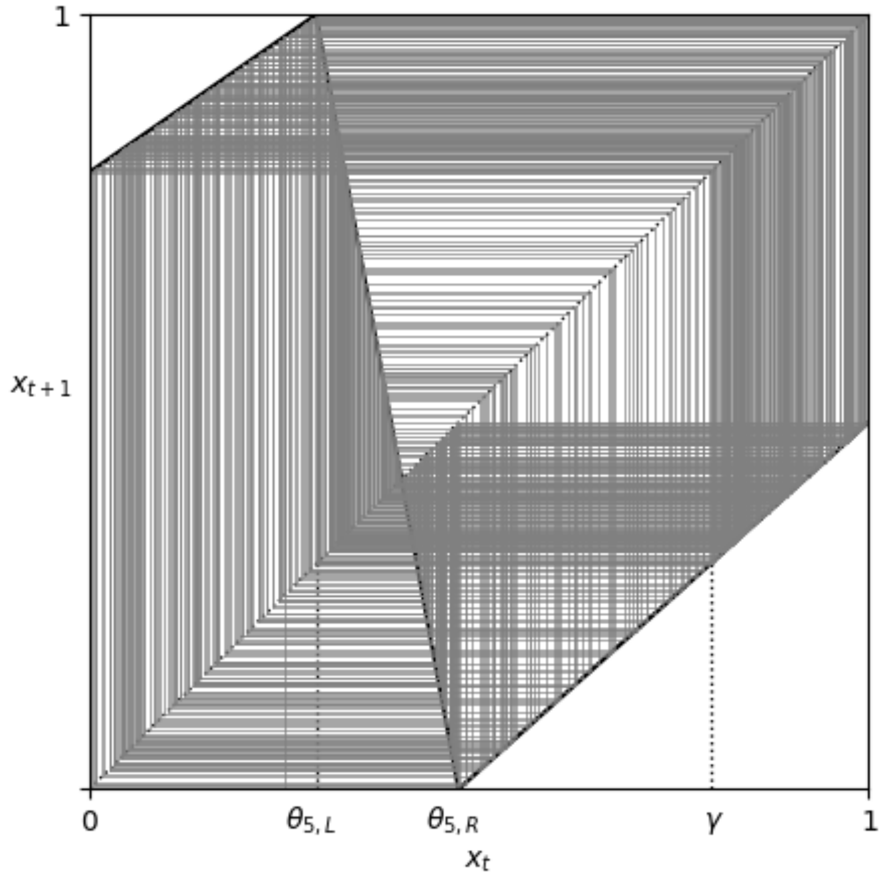

Figure 4: Period-5 Markov property and chaotic dynamics starting from a randomly chosen initial point. The parameter values are the same as in Fig. 3.

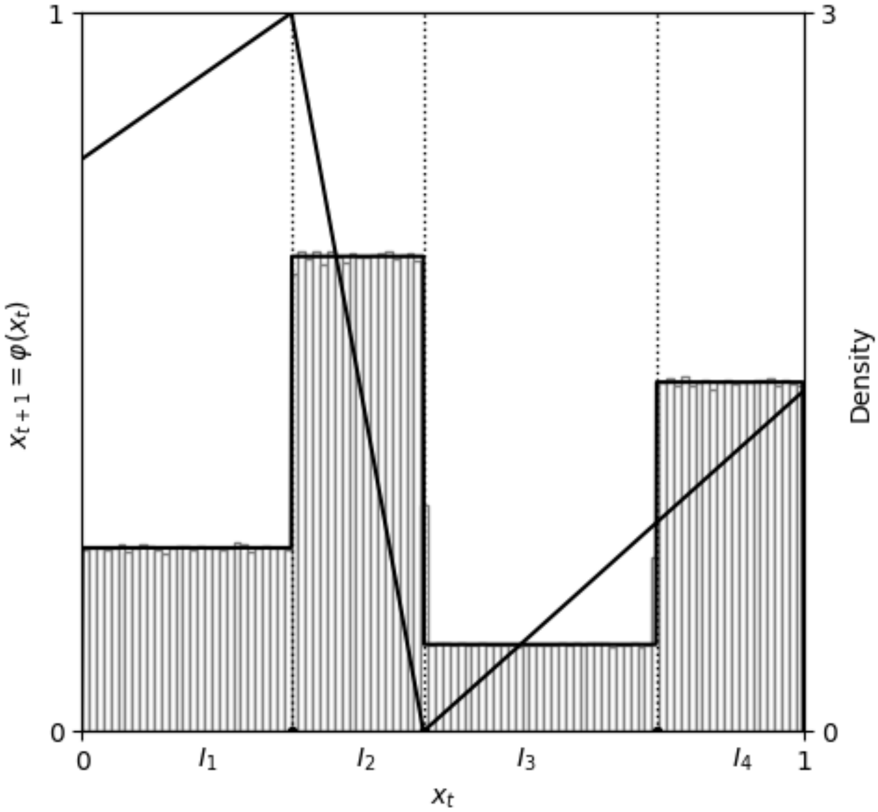

Figure 5: Theoretical and estimated invariant density for the period-5 Markov map. Density $p(x) = \pi_5^*(x)/\sum_{i=1}^4 \pi_i\,|I_i|$. Simulated histogram of 10^6 iterations. $\alpha = 0.7$, $\beta = 0.9$, $A = 20$, $B = (\alpha A)/\beta$, $\hat{\sigma} \approx 0.1075$, $\hat{s} \approx 0.372$.

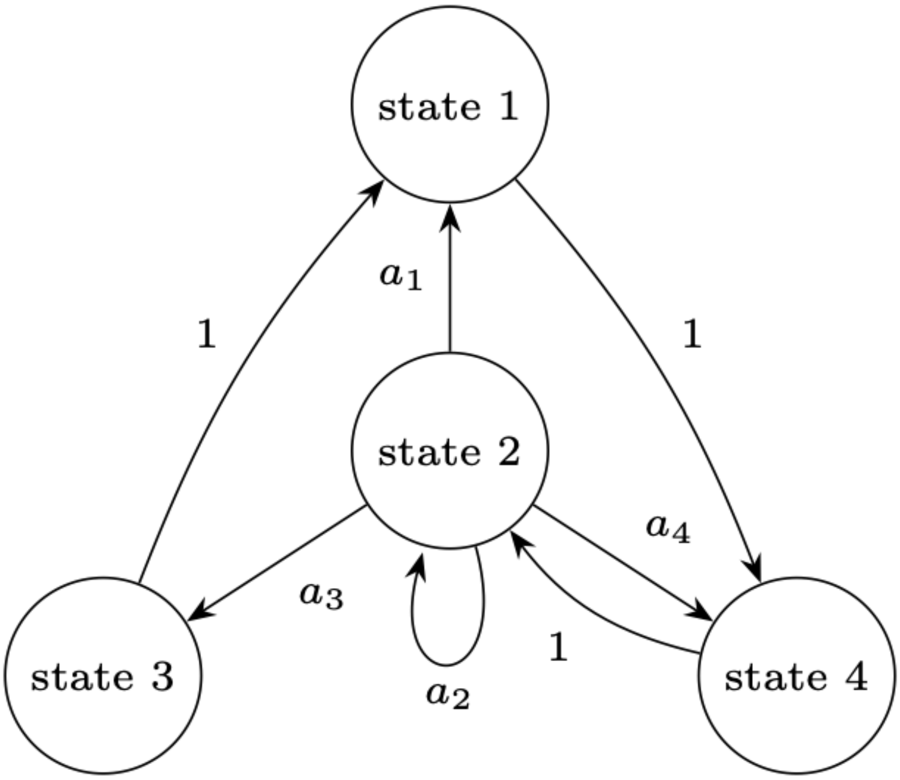

Figure 6: State transition diagram of a Markov chain corresponding to P_{φ_5} for $a_i \in (0, 1)$.

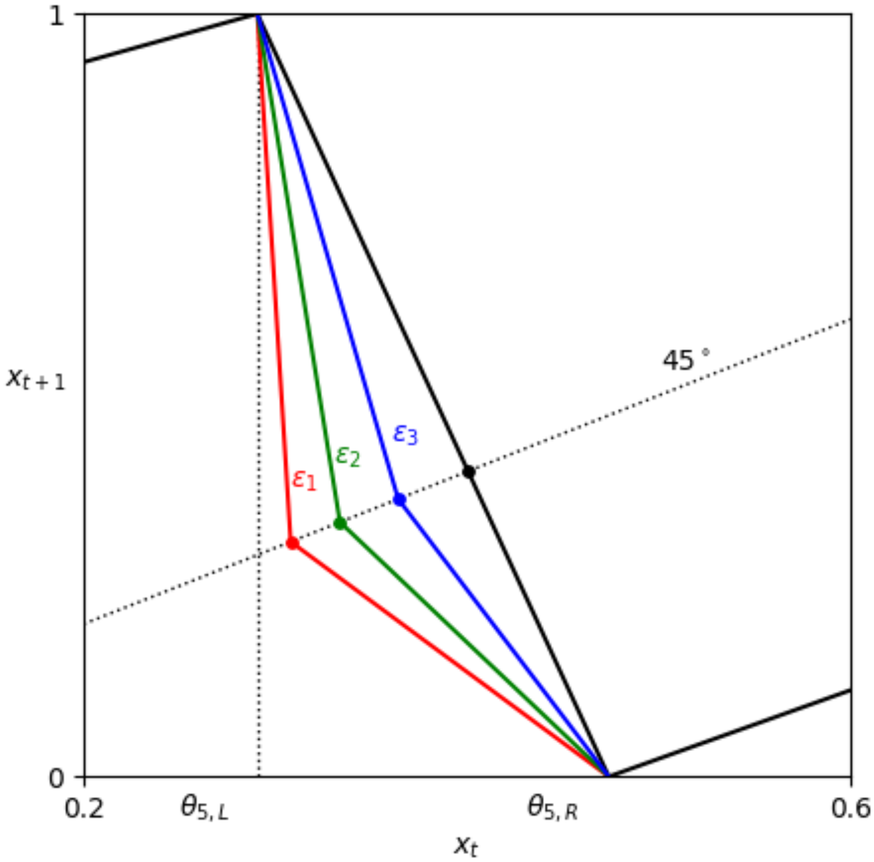

Figure 7: Perturbations for different values of ε.

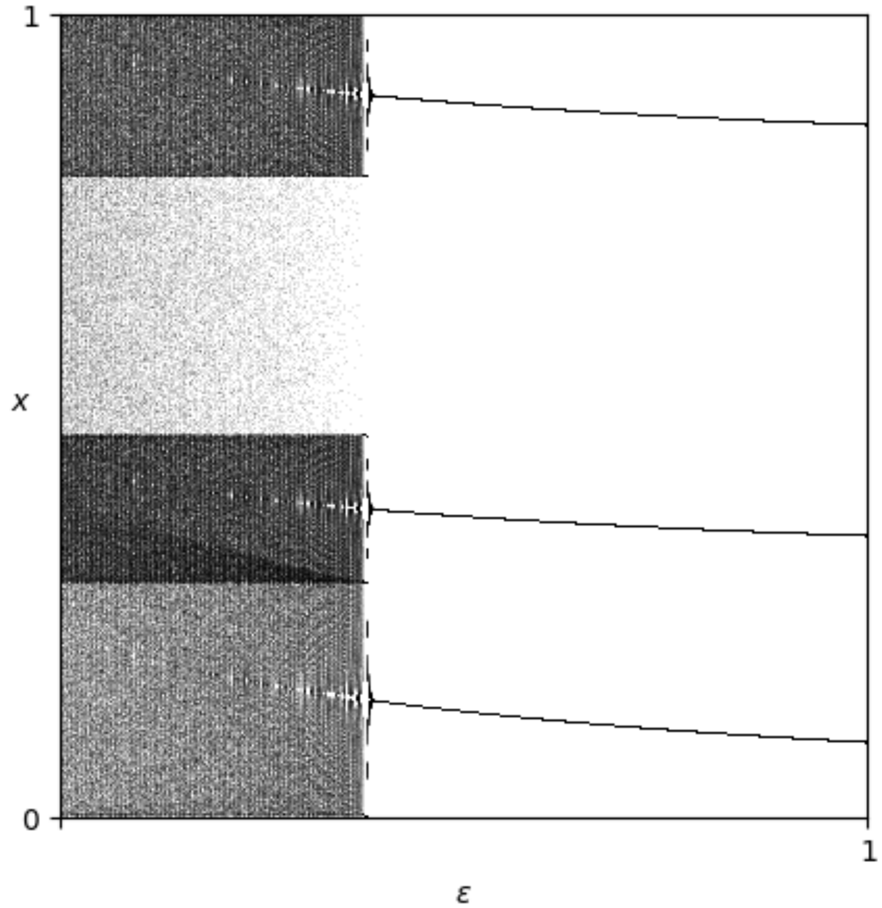

Figure 8: Bifurcation diagram with respect to ε under the period-5 Markov parameters, showing that the density is well preserved near $\varepsilon = 0$ and thus indicating robustness.