Exotic Dynamics of Inflation: Numerical Simulations

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Abstract

This paper develops a deterministic inflation model via a piecewise linear expectations-augmented Phillips curve. Simulations show that the area-preserving property of the model entails complex dynamics including non-attracting but observable periodic or chaotic behaviors. Such behaviors are relatively uncommon in the economic literature.

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1 Introduction

Inflation has always been a crucial topic in macroeconomics, as well as its relationship with the unemployment rate. In this paper, we build a simple inflation-unemployment model exhibiting complex behaviors that are not commonly observed in the economic literature. The model is composed of simple macroeconomic relations such as the expectations-augmented Phillips curve, static expectations, and a dynamic version of the Okun's law, all components of which can be found in undergraduate macroeconomic textbooks such as Blanchard (1997).

In addition to these standard relationships, we assume that the Phillips curve is *piecewise linear*. This allows us to introduce slight nonlinearity into the model with all other relationships being linear.

A similar inflation model[1] was analyzed by Soliman (1996a, 1996b), who assumes a smooth Phillips curve instead of a piecewise linear one. Nonetheless, adopting piecewise linearity has a distinct advantage over other types of nonlinearities. For example, Umezuki (2019) shows that, in the same framework as presented here, the piecewise linearity of the Phillips curve allows the direct calculations of certain exact analytical results, which could not be obtained under the smooth nonlinearity assumption. For instance, it can be shown that the model exhibits a transverse homoclinic point, which implies the existence of a horseshoe, (i.e., a chaotic invariant set). However, the same could hardly be shown for the Soliman's model without relying on numerical methods.

In this paper, we leave aside the detailed analysis and only demonstrate some numerical results for complex but less common dynamics of the model.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 shows the results of some numerical simulations for several specific sets of parameter values. Section 4 deals with the polygonal dynamics with integer parameters. Section 5 provides some concluding remarks.

2 The Model

First, we formalize four standard macroeconomic relationships as in Blanchard (1997).

$$\pi_t = f(u_t) + \pi_t^e, \tag{1}$$

[1] Yoshida (2015) employs a similar inflation model with a nonlinear Okun's law and a linear Phillips curve, and numerically shows that the model exhibits chaotic dynamics.

$$\begin{aligned} u_{t+1} &= -\beta(g_t - g_n) + u_t, \qquad \beta > 0; &(2)\\ g_t &= m - \pi_t, &(3)\\ \pi^e_{t+1} &= \pi_t. &(4) \end{aligned}$$

Eq.(1) describes the expectations-augmented Phillips curve: the gap between the actual inflation rate π_t and the expected inflation rate π^e_t is related to the unemployment rate u_t through the function f. The subscript t denotes time that discretely extends from 0 to infinity. Eq.(2) represents the dynamic version of the Okun's law. According to this formula, if the output growth rate g_t is above the natural growth rate g_n, then the time variance of unemployment decreases. Eq.(3) presents the equilibrium condition in the money market. We assume here that the nominal growth rate of money, m, is constant over time. We also assume that the expectations are static as per Eq.(4).

We call the graph of the function f the *basic Phillips curve*, because if, instead of Eq.(1), we would relate π_t to u_t directly as $\pi_t = f(u_t)$, we would obtain the Phillips curve prior to augmentation with expectations. We assume that the basic Phillips curve has a piecewise linear form represented by

$$f(u_t) = \begin{cases} -a_1 u_t + b_1 & \text{if } u_t \leq \theta \\ -a_2 u_t + b_2 & \text{if } u_t > \theta, \end{cases} \tag{5}$$

where either $a_1 > a_2 > 0$ and $b_1 > b_2 > 0$ when f is convex, or $a_2 > a_1 > 0$ and $b_2 > b_1 > 0$ when f is concave.[2] In either case, f has a kink at

$$u_t = \theta = \frac{b_1 - b_2}{a_1 - a_2}.$$

Eqs.(1)–(5) can be reduced to the following second-order difference equation in terms of unemployment:

$$u_{t+2} = (2 - \beta a_i)u_{t+1} - u_t + b_i, \tag{6}$$

where $i = 1$ if $u_{t+1} \leq \theta$ and $i = 2$ if $u_{t+1} > \theta$.

Let us introduce new variables as follows:

$$u_t = Cy_t + \theta \quad \text{and} \quad y_{t+1} = x_t$$

with

$$C = \frac{\beta(a_1 b_2 - a_2 b_1)}{a_1 - a_2}.$$

[2]There have been debates (e.g., Laxton et al., 1999) about the convexity with respect to unemployment of the basic Phillips curve for the United States. Nonetheless, this issue is not much relevant in our formulation.

The change of variables transforms Eq.(6) into the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ in the following form:

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (2 - \beta a_i)x - y + 1 \\ x \end{pmatrix}, \tag{7}$$

where $i = 1$ if $Cx \leq 0$ and $i = 2$ if $Cx > 0$.

Note that the map (7) directly depends on the parameters a_1, a_2 and β. The parameters b_1 and b_2 influence the dynamics through C, while m and g_n have no impact on the model dynamics. Moreover, the value of C itself does not matter, while its sign does. By inspection, we see that the case where f is concave and $C < 0$ is equivalent to the case where f is convex and $C > 0$. Furthermore, the case where f is concave and $C > 0$ is equivalent to the case where f is convex and $C < 0$. Thus, without loss of generality we can focus on the case where f is convex. The positivity (negativity) of C corresponds to the case where the basic Phillips curve has the kink at $(\pi_t, u_t) = (\pi^*, \theta)$ with $\pi^* > 0$ ($\pi^* < 0$). Fig.1 (a) and (b) depict examples of the basic Phillips curve when $C > 0$ and $C < 0$, respectively.

<<Insert Fig.1 around here>>

Finally, we note that the determinant of the Jacobian matrix of the map (7) is unity wherever the derivative exists. In such a case, the map is referred to as *area-preserving*. This implies that no invariant set for F can be an attractor or a repeller. See Guckenheimer and Holmes (1997) for the use of terminology.

3 Simulation examples

In this section, we present several figures, each of which plots a trajectory generated by the map (7) for a specific set of parameter values.

<<Insert Figs. 2–5 around here>>

Figs.2 and 3 illustrate the cases where $|2 - \beta a_i|$ is unity for $i = 1, 2$, which implies that all coefficients in Eq.(7) are integers. In both cases, we can find a fixed point, periodic points, and an orbit wandering on the plane in a complex manner while avoiding the *polygonal regions*. We can see a triangle and four hexagons in Fig.2 and six hexagons in Fig.3. In the next section, we explain how such polygonal regions might arise.

Figs.4 and 5 depict other complex orbits for cases with non-integer coefficients. In these cases, the polygons disappear, while elliptic structures arise. Counterintuitively, every orbit depicted in the figures represents *not* an attractor due to the area-preserving property of the map.

4 Polygonal dynamics

In Figs.2 and 3, all parameters are integers: $(a_1, a_2, \beta) = (3, 1, 1)$. In these cases, we have observed the emergence of polygonal regions on the plane. Devaney (1984) and Aharonov et al. (1997) study the case of Fig.3. Here, we investigate the case of Fig.2 in more detail. Here, the map (7) turns to

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -|x| - y + 1 \\ x \end{pmatrix}. \tag{8}$$

The map (8) has a center fixed point at $(x, y) = \left(\frac{1}{3}, \frac{1}{3}\right) \in \mathbb{R}^2$. Let J be a Jacobian matrix of the map (8). In the neighborhood of the fixed point, J is given by

$$J = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

and thus, $J^3 = \mathrm{Id}$. It is easy to see that the maximal neighborhood of $\left(\frac{1}{3}, \frac{1}{3}\right)$, on which $J^3 = \mathrm{Id}$, is a triangle with vertices at $(1, 0), (0, 1)$, and $(0, 0)$. Therefore, *all points* within the triangular region, except the fixed point, have period three. Thus, the amplitude of the period-three cycle within the triangular region depends on the initial condition.

Furthermore, the map (8) exhibits the periodic center points of period four at $(3, -1), (-1, 3), (-3, -1)$, and $(-1, -3)$. By the same argument, let J_4 be a Jacobian matrix of the fourth iteration of the map (8). In the neighborhood of the periodic point $(3, -1)$, for instance, J_4 is given by

$$J_4 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{10}$$

and thus, $J_4^6 = \mathrm{Id}$. Therefore, all points in the neighborhood of each periodic point have period 24. Again, the amplitude of this cycle depends on the initial conditions.

It is not hard to conclude that such polygonal structures will easily be destroyed when the parameters are perturbed from the given integers. Fig.6 provides an example. In this case, there are infinitely many elliptic orbits instead of the period three and 24 cycles mentioned above.

<<Insert Fig.6 around here>>

For the dynamics outside the polygons, see Umezuki (2019), which shows the presence of a chaotic invariant set.

<<Insert Fig.7 around here>>

5 Concluding remarks

Fig.7 depicts how the actual inflation rate and the unemployment rate are related to the basic Phillips curve. We can observe that the complex trajectory is scattered around the basic Phillips curve. Note that the kink of the basic Phillips curve is the only element of nonlinearity responsible for the complex dynamics in our model.

Finally, the area-preserving property of our economic system is not robust. In fact, we can show that a slight change in the expectations formation can make our model *dissipative*. As a result, some less common, less intuitive dynamic properties such as polygonal and elliptic structures, and some sort of sensitivity to initial conditions would easily disappear.

References

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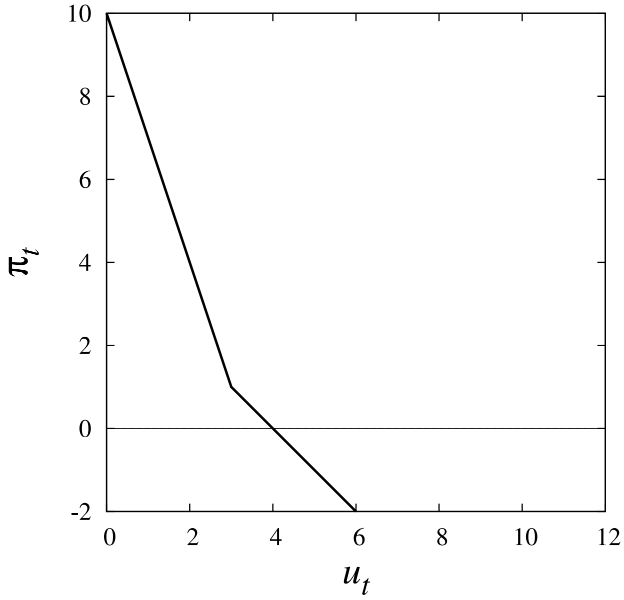
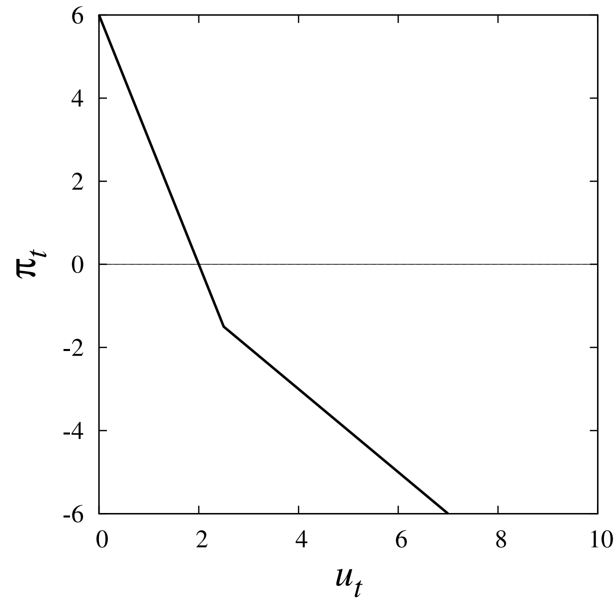


(a) The basic Phillips curve represented by Eq.(5). $a_1 = 3, a_2 = 1, b_1 = 10$ and $b_2 = 4$.



(b) The basic Phillips curve represented by Eq.(5). $a_1 = 3, a_2 = 1, b_1 = 6$ and $b_2 = 1$.

Figure 1: Fig.(a): $C > 0$. Fig.(b): $C < 0$.

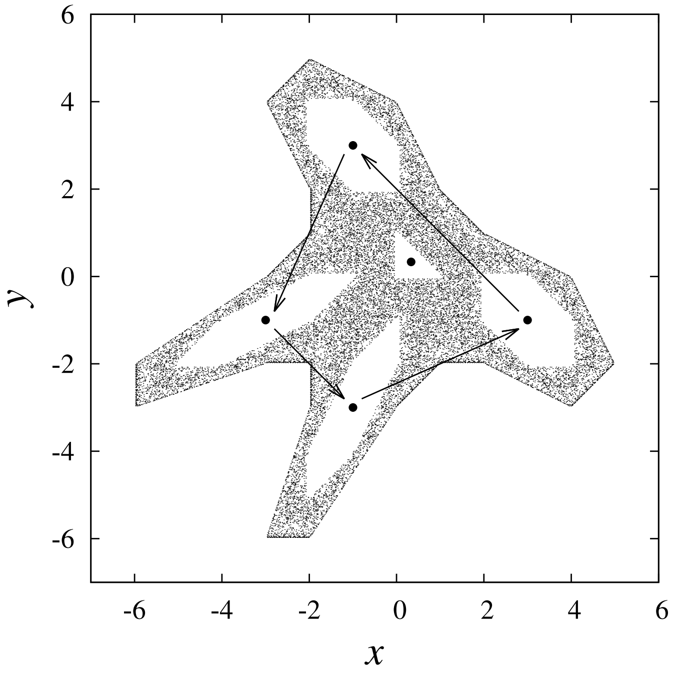


Figure 2: $a_1 = 3, a_2 = 1, \beta = 1$, and $C < 0$.

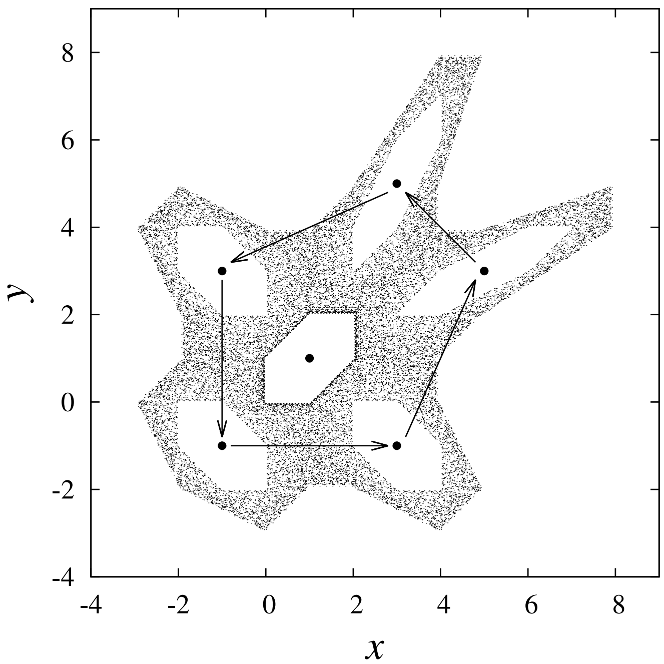


Figure 3: $a_1 = 3, a_2 = 1, \beta = 1$, and $C > 0$.

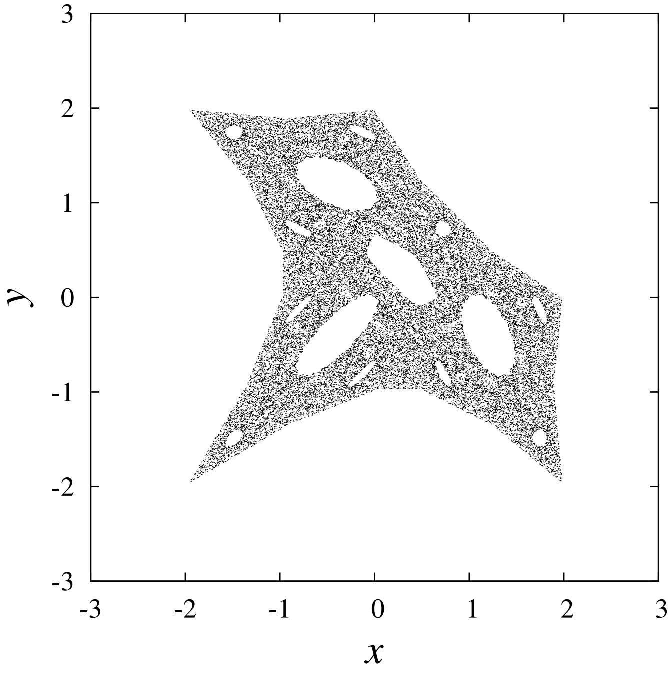


Figure 4: $a_1 = 3.5, a_2 = 1.5, \beta = 1$, and $C < 0$.

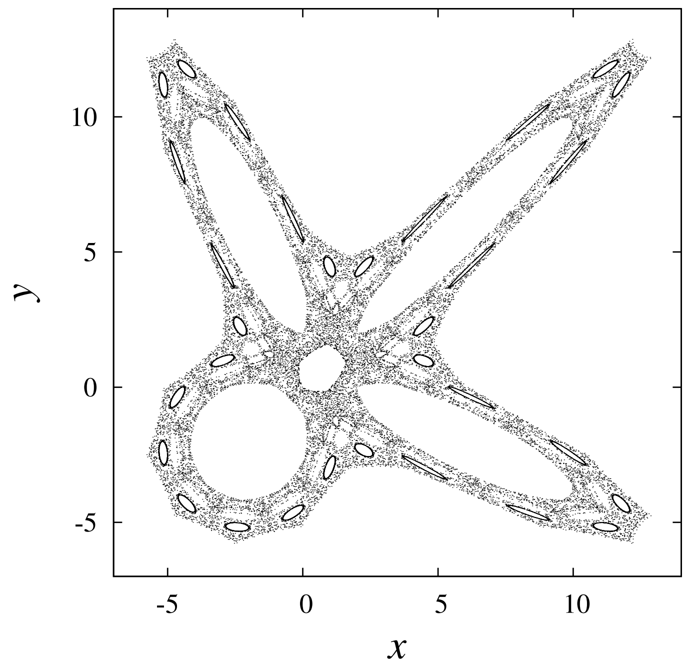
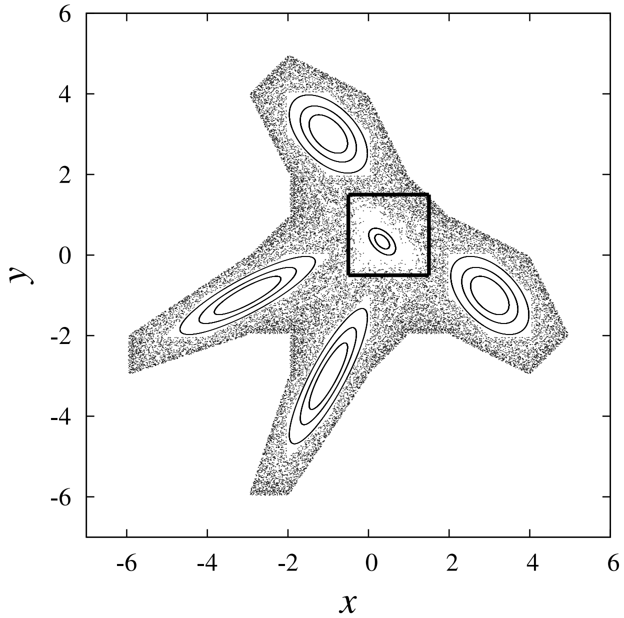
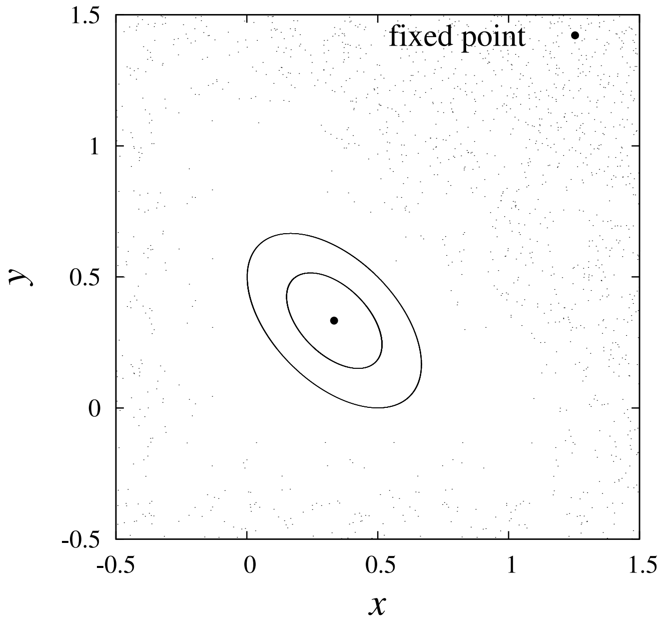


Figure 5: $a_1 = 3.5, a_2 = 1.5, \beta = 1$, and $C > 0$.



(a) $a_1 = 3.001, a_2 = 1, \beta = 1$ and $C < 0$.



(b) Closer look at the square in (a).

Figure 6: Fig.(a) depicts a complex orbit and some elliptic orbits generated by the map slightly perturbed from the case of Fig.2. Fig.(b) depicts an enlargement of the neighborhood of the fixed point.

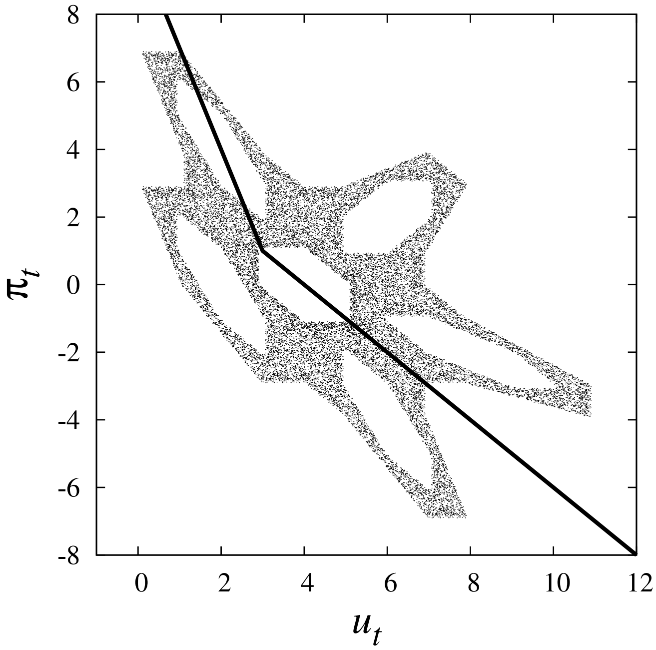


Figure 7: A trajectory of the actual inflation rate and unemployment rate, which corresponds to Fig.3. The basic Phillips curve given by Eq.(5) is superimposed. $a_1 = 3, a_2 = 1, b_1 = 10, b_2 = 4$ and $\beta = m = g_n = 1$.